

COCHORDAL ZERO DIVISOR GRAPHS AND BETTI NUMBERS OF THEIR EDGE IDEALS

LE XUAN DUNG AND THANH VU

ABSTRACT. We associate a sequence of positive integers, termed the type sequence, with a cochordal graph. Using this type sequence, we compute all graded Betti numbers of its edge ideal. We then classify all positive integer n such that the zero divisor graph of $\mathbb{Z}/n\mathbb{Z}$ is cochordal and determine all the graded Betti numbers of its edge ideal.

1. INTRODUCTION

Rather, Imran, and Pirzada [RIP] recently provided formulae for the Betti numbers of the edge ideals of zero divisor graphs of $\mathbb{Z}/n\mathbb{Z}$, denoted by $\Gamma(\mathbb{Z}_n)$ when n is of the form p^4 , p^2q , and pqr where p, q, r are distinct prime numbers. These zero divisor graphs are all cochordal. Equivalently, by the result of Fröberg [F], their edge ideals have a linear free resolution. Motivated by this result, we first classify all n for which the zero divisor graph of $\mathbb{Z}/n\mathbb{Z}$ is cochordal.

Theorem 1.1. *Let $n \geq 2$ be a positive integer. Then $\Gamma(\mathbb{Z}_n)$ is cochordal if and only if n is one of the following forms:*

- (1) $n = p^a$,
- (2) $n = p^a q$,
- (3) $n = pqr$,

where p, q, r are distinct prime numbers and a is a positive integer.

We then associate a new invariant to each cochordal graph, called the type sequence. Using the type sequence of a cochordal graph, we compute all the graded Betti numbers of its edge ideal. Consequently, we derive formulae for the Betti numbers of the edge ideals of zero divisor graphs of $\mathbb{Z}/n\mathbb{Z}$ when n is of the form $n = p^a$, $n = p^a q$, or $n = pqr$ where p, q, r are distinct prime numbers. Let us now introduce these concepts in more detail.

Beck [B] introduced the zero divisor graph $\Gamma(R)$ of a commutative ring R and studied the finitenistic of the colorings of $\Gamma(R)$ and its algebraic consequences. The structure of zero divisor graphs was furthered analyzed and developed by Anderson and Livingston [AL] and Mulay [M] among others. They particularly focused on a smaller induced subgraph defined by the equivalence classes of zero divisors of R ,

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known as the compressed zero divisor graph of R . Spiroff and Wickham [SW] carried this study further to investigate the associated primes of R . We will build on this idea further in our study.

Definition 1.2. Let R be a commutative ring with unity. Denote by $Z(R)$ the set of zero divisors of R . The zero divisor graph of R , denoted by $\Gamma(R)$ is a graph on the vertex set $Z(R) \setminus \{0\}$ and $\{u, v\}$ is an edge of $\Gamma(R)$ if $uv = 0$.

Arunkumara, Cameron, Kavaskar, and Tamizh Chelvam [ACKT] recently proved that zero divisor graphs are universal, even when R is restricted to boolean rings, or the rings of integers modulo n . In other words, for any finite graph G , there exists a positive integer n such that G is an induced subgraph of $\Gamma(\mathbb{Z}_n)$. As a first step toward understand the homological invariants of edge ideals of $\Gamma(\mathbb{Z}_n)$, we classify all n for which Γ_n is cochordal.

Definition 1.3. A simple graph G is called a chordal graph if every cycle of length at least 4 in G has a chord. G is cochordal if its complement $\overline{G}$ is chordal.

Let G be a simple graph on a vertex set $V(G)$ and edge set $E(G)$. A subset $U \subseteq V(G)$ is called a vertex cover of G if for every edge $\{u, v\} \in E(G)$ either u or v belongs to U . We denote by $K_{U,V}$ the complete bipartite graph with a bipartition $V(G) = U \cup V$. When $U = \{u\}$ we also denote $K_{U,V}$ by $K_{u,V}$. Motivated by the Dirac's theorem [Di] on chordal graphs, we define

Definition 1.4. Let $\mathbf{P} = (u_k, U_k, u_{k-1}, U_{k-1}, \dots, u_1, U_1)$ be an ordered set of vertices u_i and subsets $U_i \subseteq V$ of a set of vertices V such that $u_i \notin U_j$ for $j \leq i$. We define recursively the graphs G_i as follows

- (1) $G_1 = K_{u_1, U_1}$;
- (2) $V(G_j) = V(G_{j-1}) \cup V(K_{u_j, U_j})$, $E(G_j) = E(G_{j-1}) \cup E(K_{u_j, U_j})$.

We call $\mathbf{P}$ a cochordal constructible system of a graph G if $G = G_k$ and U_j is a vertex cover of G_{j-1} for all $j = 2, \dots, k$. The type of $\mathbf{P}$ is defined by $\text{type}(\mathbf{P}) = (a_k, \dots, a_1)$ where $a_j = |U_j|$ for all $j = 1, \dots, k$.

Theorem 1.5. A graph G is cochordal if and only if it has a cochordal constructible system $\mathbf{P}$.

Definition 1.6. The type of a cochordal constructible system $\mathbf{P}$ of a cochordal graph G is also called a type of G .

The notion of type of a cochordal graph is motivated from the work of Corso and Nagel [CN1, CN2]. For example, the Ferrers graph G_λ associated with a partition λ has type λ . The following is a generalization of the results of Corso and Nagel to arbitrary cochordal graphs.

Theorem 1.7. The Betti numbers of the edge ideal of a cochordal graph of type $(a_k, \dots, a_1)$ is given by

$$\beta_i(S/I) = \binom{a_k}{i} + \binom{a_{k-1} + 1}{i} + \dots + \binom{a_1 + k - 1}{i} - \binom{k}{i+1}$$

for all $i \geq 1$.

We then apply to compute all the Betti numbers of $\Gamma(\mathbb{Z}_n)$ when n is of the form $n = p^a$, $n = p^a q$, or $n = pqr$.

Theorem 1.8. *Let $a \geq 2$ be a positive integer and p be a prime number. Then $I = I(\Gamma(\mathbb{Z}_{p^a}))$ has a linear free resolution and*

$$\beta_i(S/I) = \sum_{j=\lceil \frac{a}{2} \rceil}^{a-1} p^{a-j-1}(p-1) \binom{p^j - 2}{i},$$

for all $i \geq 1$. In particular, $\text{pd}(S/I) = p^{a-1} - 2$.

Theorem 1.9. *Let a be a positive integer and p, q be two distinct prime numbers. Then $I = I(\Gamma(\mathbb{Z}_{p^a q}))$ has a linear free resolution and*

$$\beta_i(S/I) = \sum_{j=\lceil \frac{a}{2} \rceil}^{a-1} p^{a-j-1}(p-1) \binom{qp^j - 2}{i} + \sum_{j=0}^{\lceil \frac{a}{2} \rceil - 1} p^{a-j-1}(p-1) \binom{(q-1)p^j + p^{a-j} - 2}{i},$$

for all $i \geq 1$. In particular,

$$\text{pd}(S/I) = \begin{cases} qp^{a-1} - 2 & \text{if } a \geq 2, \\ q + p - 3 & \text{if } a = 1. \end{cases}$$

Theorem 1.10. *Let $p < q < r$ be prime numbers. Then $I = I(\Gamma(\mathbb{Z}_{pqr}))$ has a linear free resolution and*

$$\beta_i(S/I) = (p-1) \binom{qr + p - 3}{i} + (q-1) \binom{pr + q - 3}{i} + (r-1) \binom{pq + r - 3}{i},$$

for all $i \geq 1$. In particular, $\text{pd}(S/I) = qr + p - 3$

In Section 2, we discuss cochordal constructible systems and prove Theorem 1.5. We then prove Theorem 1.7 and deduce its consequences. In Section 3, we establish Theorem 1.1 and compute all the Betti numbers of edge ideals of zero divisor graphs of $\mathbb{Z}/n\mathbb{Z}$ when they are cochordal.

2. COCHORDAL CONSTRUCTIBLE SYSTEMS

In this section, we introduce the notion of cochordal constructible system and prove that any cochordal graph can be constructed from a cochordal constructible system. From that, we deduce a formula for all the Betti numbers of edge ideals of cochordal graphs.

We fix the following notation throughout the paper. Assume that $S = k[x_1, \dots, x_n]$ is a standard graded polynomial ring over a field k with the graded maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. An ideal P generated by variables of S is called a monomial prime ideal. If $P = (x_{i_1}, \dots, x_{i_s})$, we denote by $\text{supp } P = \{i_1, \dots, i_s\}$.

Let $I = (f_1, \dots, f_t)$ be a monomial ideal of S and y be a variable of S . Let $J = (f_j \mid y \text{ does not divide } f_j)$. Then I has a unique decomposition, called the y -partition of I , $I = J + yL$, where $yL = (f_j \mid y \text{ divides } f_j)$.

2.1. Betti numbers. Let M be a finitely generated graded S -module. For integers i, j with $i \geq 0$, the i -th Betti number of M in degree j is

$$\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(k, M)_j.$$

The projective dimension of M , denoted by $\operatorname{pd}_S(M)$ and regularity of M , denoted by $\operatorname{reg}_S(M)$ are defined by

$$\begin{aligned} \operatorname{pd}_S(M) &= \sup\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\}, \\ \operatorname{reg}_S(M) &= \sup\{j - i \mid \beta_{i,j}(M) \neq 0\}. \end{aligned}$$

The following result is well-known.

Lemma 2.1. *Let x be a variable and I a nonzero homogeneous ideal of S . Then*

$$\beta_{i,j}(xI) = \beta_{i,j-1}(I) \text{ for all } i \geq 0.$$

2.2. Betti splittings. Betti splittings were introduced by Francisco, Ha, and Van Tuyl [FHV] for monomial ideals. We follow the treatment of Betti splittings in [NV]. Let P, I, J be proper homogeneous ideals of S such that $P = I + J$.

Definition 2.2. The decomposition $P = I + J$ is called a Betti splitting if for all $i \geq 0$, the following equality of Betti numbers holds: $\beta_i(P) = \beta_i(I) + \beta_i(J) + \beta_{i-1}(I \cap J)$.

Lemma 2.3. *Let I be a quadratic monomial ideal and x be a variable. Then the x -partition of I , $I = xP + J$ is a Betti splitting.*

Proof. Since I is generated by quadratic monomials, P is generated by variables. The conclusion follows from [NV, Corollary 4.12]. $\square$

2.3. Cochordal constructible systems. Let G be a simple graph on vertex set $V(G) = [n]$ and edge set $E(G) \subseteq V(G) \times V(G)$. The edge ideal of G is defined by

$$I(G) = (x_i x_j \mid \{i, j\} \text{ is an edge of } G) \subset S.$$

The following is an essential property of cochordal graphs.

Lemma 2.4. *Let $I = I(G)$ be the edge ideal of a cochordal graph. Then there exists a variable x_i such that $I : x_i = P$ is a monomial prime ideal. In other words, $I = x_i P + J$ and $J \subset P$.*

Proof. The conclusion follows from [JV, Proposition 3.19]. $\square$

Proof of Theorem 1.5. First, assume that G is cochordal. We prove by induction on n that G has a cochordal constructible system. The base case $n = 2$ is clear. Now, assume that $n \geq 3$. By Lemma 2.4, there exists a variable x_i such that $I(G) : x_i = P$ is a monomial prime ideal. Hence, $U = \operatorname{supp} P$ is a vertex cover of G' , the induced subgraph of G on $V(G) \setminus \{i\}$. By induction, G' has a cochordal constructible

system $\mathbf{P}' = (u_{k-1}, U_{k-1}, \dots, u_1, U_1)$. Set $u_k = i$ and $P_k = U$, we deduce that $\mathbf{P} = (u_k, U_k, \dots, u_1, U_1)$ is a cochordal constructible system of G .

Now, assume that G has a cochordal constructible system $\mathbf{P}$. We prove by induction on k that G is cochordal. By the result of Fröberg [F], it suffices to prove that $\text{reg}(I(G)) = 2$. By definition, $G = G_k$. The base case $k = 1$ is clear. By induction G_{k-1} is cochordal and $I(G_k) = x_k P_k + I(G_{k-1})$ where x_k corresponds to the vertex u_k and P_k is the monomial prime ideal generated by variables corresponding to U_k . By Lemma 2.3, the decomposition $I(G_k) = x_k P_k + I(G_{k-1})$ is a Betti splitting. Furthermore, by assumption, $x_k P_k \cap I(G_{k-1}) = x_k I(G_{k-1})$. Hence,

$$\text{reg}(I(G_k)) = \max\{\text{reg}(I(G_{k-1})), \text{reg}(x_k P_k), \text{reg}(x_k I(G_{k-1})) - 1\} = 2.$$

The conclusion follows. $\square$

2.4. Betti numbers of cochordal graphs. Let G be a cochordal graph of type $(a_k, \dots, a_1)$. By Theorem 1.5, the edge ideal of G can be written as

$$I(G) = x_k P_k + x_{k-1} P_{k-1} + \dots + x_1 P_1$$

where P_j is a monomial prime ideal generated by a_j variables. Let $I_j = x_j P_j + \dots + x_1 P_1$ for all $j = 1, \dots, k$.

Proof of Theorem 1.7. We prove by induction on k . The case $k = 1$ is clear as $I_1 = x_1 P_1$ with P_1 is a monomial prime ideal generated by a_1 variables.

Now, assume that the statement holds for $k-1$. Note that, $\beta_1(S/I) = a_k + \dots + a_1 = a_k + (a_{k-1} + 1) + \dots + (a_1 + k - 1) - \binom{k}{2}$. Thus, we may assume that $i \geq 2$. By the proof of Theorem 1.5, the decomposition $I_k = x_k P_k + I_{k-1}$ is a Betti splitting and $x_k P_k \cap I_{k-1} = x_k I_{k-1}$. Hence, we have for all $i \geq 2$,

$$\beta_i(S/I_k) = \beta_i(S/I_{k-1}) + \beta_{i-1}(S/I_{k-1}) + \beta_i(S/(x_k P_k)).$$

By induction, we have

$$\begin{aligned} \beta_i(S/I_{k-1}) &= \binom{a_{k-1}}{i} + \dots + \binom{a_1 + k - 2}{i} - \binom{k-1}{i+1} \\ \beta_{i-1}(S/I_{k-1}) &= \binom{a_{k-1}}{i-1} + \dots + \binom{a_1 + k - 2}{i-1} - \binom{k-1}{i}. \end{aligned}$$

The conclusion follows from Lemma 2.1, the fact that $\beta_i(S/P_k) = \binom{a_k}{i}$, and the binomial identity $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$. $\square$

Dochtermann [Doc] gave a formula for the Betti numbers of the ideal of the complement of a d -chordal cluster. His formula requires the computation of the maximal clique sizes containing the exposed edge at each step. Generally, our method is computationally simpler, as it depends on the number of variables, whereas Dochtermann's formula depends on the number of generators of I .

Example 2.5. Consider the following edge ideal

$$I = x_3(x_1, x_2) + x_2(x_1, x_4) + x_1(x_5, x_6) \subseteq R = k[x_1, \dots, x_6].$$

According to Dochtermann's formula, we see that the sequence of exposed edges are $x_1x_6, x_1x_5, x_1x_2, x_1x_3, x_2x_3, x_2x_4$. Hence, the k_j -invariants are $0, 1, 2, 3, 1, 2$. In our formula, I has type $(2, 2, 2)$. Also, I can be rewritten as

$$I = x_1(x_2, x_3, x_5, x_6) + x_2(x_3, x_4).$$

Thus, I also has type $(4, 2)$. By Theorem 1.7, the Betti table of I is

	0	1	2	3
—	—	—	—	—
2	6	9	5	1

Corollary 2.6. *Let G be a cochordal graph of type $(a_k, \dots, a_1)$. Then $\text{pd}(S/I(G)) = \max\{a_k, a_{k-1} + 1, \dots, a_1 + k - 1\}$.*

Proof. The conclusion follows from Theorem 1.7. □

Remark 2.7. Antonino Ficarra pointed out that Lemma 2.4 holds more generally for quadratic monomial ideals having linear free resolutions [Fi, Lemma 2.4]. Hence, we can extend our concept of type sequence to any quadratic monomial ideal with a linear free resolution. This fact also follows from taking the polarization. In particular, Theorem 1.7 and Corollary 2.6 hold for arbitrary quadratic monomial ideals having a linear free resolution.

3. COCHORDAL ZERO DIVISOR GRAPHS

This section is devoted to proving Theorem 1.1 and computing Betti numbers of edge ideals of zero divisor graphs of $\mathbb{Z}/n\mathbb{Z}$ when n is of the form $n = p^a$, $n = p^a q$, or $n = pqr$ where p, q, r are distinct prime numbers and a is a positive integer. We first recall the definition of compressed zero divisor graphs.

Let R be a commutative ring with unity. We denote by $Z^*(R)$ the set of nonzero zero divisors of R . For $x \in R$, we denote by $[x]$ the set of equivalence class of all y in R such that $\text{ann}(x) = \text{ann}(y)$.

Definition 3.1. The compressed zero divisor graph of R , denoted by $\Gamma_E(R)$, is a graph on the vertex set consisting of equivalence classes of elements in $Z^*(R)$ and two distinct equivalence classes $[x]$ and $[y]$ are joined by an edge if and only if $[x] \cdot [y] = [0]$.

We now prove that when $n = p^a$, $n = p^a q$, or $n = pqr$, $\Gamma(R)$ is cochordal and compute its type sequence. Each element of $R = \mathbb{Z}/n\mathbb{Z}$ has a representative i such that $0 \leq i \leq n - 1$. Thus, we can view the vertex set of $\Gamma(R)$ as a subset of $\{1, \dots, n - 1\}$. We order the vertices of $\Gamma(R)$ within the same equivalence class by the natural order on $\mathbb{Z}$. We also use the following convention: if $\mathbf{s}_1, \dots, \mathbf{s}_k$ are sequences, then $\mathbf{s} = (\mathbf{s}_k, \dots, \mathbf{s}_1)$ denotes the concatenation of the sequences $\mathbf{s}_k, \dots, \mathbf{s}_1$.

Lemma 3.2. *Let p be a prime number and a be a positive integer at least 2. Let*

$$\mathbf{s}_i = (p^i - p^{a-i-1} - 1, p^i - p^{a-i-1} - 2, \dots, p^i - p^{a-i}) \text{ for } i = \lceil \frac{a}{2} \rceil, \dots, a-1.$$

Then $\Gamma(\mathbb{Z}_{p^a})$ is cochordal and has a type $\mathbf{s} = (\mathbf{s}_{a-1}, \dots, \mathbf{s}_{\lceil \frac{a}{2} \rceil})$.

Proof. Let $n = p^a$ and $R = \mathbb{Z}/n\mathbb{Z}$. For each $i = 1, \dots, a-1$, we denote by

$$V_i = \{x \in R \mid [x] = [p^i]\}.$$

First, we have $|V_i| = \varphi(p^{a-i}) = \varphi_i = p^{a-i-1}(p-1)$, where $\varphi(n)$ is the Euler's function. We assume that $V_i = \{v_{i,1}, \dots, v_{i,\varphi_i}\}$ and $v_{i,j} < v_{i,j+1}$ for all i and j . We now define

$$U_{i,j} = V_{i-1} \cup V_{i-2} \cup \dots \cup V_{a-i} \cup \{v_{i,1}, \dots, v_{i,j-1}\}$$

for $i = \lceil \frac{a}{2} \rceil, \dots, a-1$ and $j = 1, \dots, \varphi_i$. Note that $\{v_{i_1,j_1}, v_{i_2,j_2}\}$ is an edge of Γ_n if and only if $i_1 + i_2 \geq a$. Hence, the ordered set $\mathbf{P} = (v_{i,j}, U_{i,j} \mid i = a-1, \dots, \lceil \frac{a}{2} \rceil, j = \varphi_i, \dots, 1)$ is a cochordal constructible system of $\Gamma(\mathbb{Z}_n)$. By Theorem 1.5, $\Gamma(\mathbb{Z}_n)$ is cochordal. Finally, we have

$$\begin{aligned} |U_{i,j}| &= |V_{i-1}| + \dots + |V_{a-i}| + j - 1 \\ &= p^{a-i}(p-1)(1 + \dots + p^{2i-a-1}) + j - 1 \\ &= p^i - p^{a-i} + j - 1, \end{aligned}$$

for all $i = \lceil \frac{a}{2} \rceil, \dots, a-1$ and $j = 1, \dots, \varphi_i$. The conclusion follows. $\square$

Lemma 3.3. *Let p, q be distinct prime numbers and a be a positive integer. Let*

$$\begin{aligned} \mathbf{s}_i &= (qp^i - p^{a-i-1} - 1, qp^i - p^{a-i-1} - 2, \dots, qp^i - p^{a-i}) \text{ for } i = \lceil \frac{a}{2} \rceil, \dots, a-1, \\ \mathbf{s}_i &= ((q-1)p^i + p^{a-i-1}(p-1) - 1, (q-1)p^i + p^{a-i-1}(p-1) - 2, \dots, (q-1)p^i) \\ &\text{for } i = 0, \dots, \lceil \frac{a}{2} \rceil - 1. \end{aligned}$$

Then $\Gamma(\mathbb{Z}_{p^a q})$ is cochordal and has a type $\mathbf{s} = (\mathbf{s}_{a-1}, \dots, \mathbf{s}_0)$.

Proof. Let $n = p^a q$ and $R = \mathbb{Z}/n\mathbb{Z}$. We set

$$\begin{aligned} U_i &= \{x \in R \mid [x] = [p^i]\} \text{ for } i = 1, \dots, a, \\ V_i &= \{x \in R \mid [x] = [p^i q]\} \text{ for } i = 0, \dots, a-1. \end{aligned}$$

Then, $|U_a| = q-1$ and

$$\begin{aligned} |U_i| &= \varphi(p^{a-i} q) = p^{a-i-1}(p-1)(q-1) \text{ for } i = 1, \dots, a-1 \\ |V_i| &= \varphi(p^{a-i}) = \varphi_i = p^{a-i-1}(p-1) \text{ for } i = 0, \dots, a-1. \end{aligned}$$

We assume that $V_i = \{v_{i,1}, \dots, v_{i,\varphi_i}\}$ and $v_{i,j} < v_{i,j+1}$ for all i and j . We now define

$$W_{i,j} = U_a \cup U_{a-1} \cup \dots \cup U_{a-i} \cup V_{i-1} \cup \dots \cup V_{a-i} \cup \{v_{i,1}, \dots, v_{i,j-1}\}$$

for $i = 0, \dots, a-1$ and $j = 1, \dots, \varphi_i$. For $x \in U_i$ and $y \in V_j$, we have $\{x, y\}$ is an edge of Γ_n if and only if $i+j \geq a$. For $x \in V_i$ and $y \in V_j$, we have $\{x, y\}$ is an edge of Γ_n if and

only if $i + j \geq a$. Hence, the ordered set $\mathbf{P} = (v_{i,j}, W_{i,j} \mid i = a-1, \dots, 0, j = \varphi_i, \dots, 1)$ is a cochordal constructible system of $\Gamma(\mathbb{Z}_n)$. By Theorem 1.5, $\Gamma(\mathbb{Z}_n)$ is cochordal. By the proof of Lemma 3.2, we have

$$\begin{aligned} |W_{i,j}| &= |U_a| + \dots + |U_{a-i}| + |V_{i-1}| + \dots + |V_{a-i}| + j - 1 \\ &= (q-1) + (q-1)(p-1)(1 + \dots + p^{i-1}) + p^i - p^{a-i} + j - 1 \\ &= (q-1)p^i + p^i - p^{a-i} + j - 1 \\ &= qp^i - p^{a-i} + j - 1, \end{aligned}$$

for all $i = \lceil \frac{a}{2} \rceil, \dots, a-1$ and $j = 1, \dots, \varphi_i$. For $i = 0, \dots, \lceil \frac{a}{2} \rceil - 1$ and $j = 1, \dots, \varphi_i$, we have

$$\begin{aligned} |W_{i,j}| &= |U_a| + \dots + |U_{a-i}| + j - 1 \\ &= (q-1)p^i + j - 1. \end{aligned}$$

The conclusion follows. $\square$

Lemma 3.4. *Let p, q, r be distinct prime numbers. Let*

$$\begin{aligned} \mathbf{s}_1 &= (r + (p-1)(q-1) - 2, \dots, (p-1)(q-1)) \\ \mathbf{s}_2 &= (pr - p + q - 2, \dots, pr - p) \\ \mathbf{s}_3 &= (qr + p - 3, \dots, qr - 1). \end{aligned}$$

Then $\Gamma(\mathbb{Z}_{pqr})$ is cochordal and has a type $\mathbf{s} = (\mathbf{s}_3, \mathbf{s}_2, \mathbf{s}_1)$.

Proof. Let $n = pqr$ and $R = \mathbb{Z}/n\mathbb{Z}$. We assume that $p < q < r$ and set

$$\begin{aligned} U_1 &= \{x \in R \mid [x] = [p]\}, U_2 = \{x \in R \mid [x] = [q]\}, U_3 = \{x \in R \mid [x] = [r]\} \\ V_1 &= \{x \in R \mid [x] = [pq]\}, V_2 = \{x \in R \mid [x] = [pr]\}, V_3 = \{x \in R \mid [x] = [qr]\}. \end{aligned}$$

Let $\varphi_1 = |V_1| = r-1$, $\varphi_2 = |V_2| = q-1$, and $\varphi_3 = |V_3| = p-1$. Write $V_i = \{v_{i,1}, \dots, v_{i,\varphi_i}\}$ and $v_{i,j} < v_{i,j+1}$ for all $i = 1, 2, 3$ and $j = 1, \dots, \varphi_i$. We now define

$$\begin{aligned} W_{1,j} &= U_3 \cup \{v_{1,1}, \dots, v_{1,j-1}\} \text{ for } j = 1, \dots, \varphi_1 \\ W_{2,j} &= V_1 \cup U_2 \cup \{v_{2,1}, \dots, v_{2,j-1}\} \text{ for } j = 1, \dots, \varphi_2 \\ W_{3,j} &= V_1 \cup V_2 \cup U_1 \cup \{v_{3,1}, \dots, v_{3,j-1}\} \text{ for } j = 1, \dots, \varphi_3. \end{aligned}$$

Then the ordered set $\mathbf{P} = (v_{i,j}, W_{i,j} \mid i = 3, 2, 1, j = \varphi_i, \dots, 1)$ is a cochordal constructible system of $\Gamma(\mathbb{Z}_n)$. By Theorem 1.5, $\Gamma(\mathbb{Z}_n)$ is cochordal. Finally, we have

$$\begin{aligned} |W_{1,j}| &= |U_3| + j - 1 = (p-1)(q-1) + j - 1 \text{ for } j = 1, \dots, \varphi_1, \\ |W_{2,j}| &= |V_1| + |U_2| + j - 1 = pr - p + j - 1 \text{ for } j = 1, \dots, \varphi_2, \\ |W_{3,j}| &= |V_1| + |V_2| + |U_1| + j - 1 = qr + j - 2 \text{ for } j = 1, \dots, \varphi_3. \end{aligned}$$

The conclusion follows. $\square$

We are now ready for

Proof of Theorem 1.1. First, we assume that $G = \Gamma(\mathbb{Z}_n)$ is cochordal. In particular, its induced matching number is 1. We divide the proof into several steps.

Step 1. n has at most 3 prime factors. Assume by contradiction that n has more than 3 prime factors, say $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} m$ where m is a positive integer and $p_1, \dots, p_4$ are distinct prime numbers not dividing m . Let $x = p_1^{a_1} p_2^{a_2} m$, $y = p_3^{a_3} p_4^{a_4} m$, $u = p_1^{a_1} p_3^{a_3} m$ and $v = p_2^{a_2} p_4^{a_4} m$. Then $\{x, y\}$ and $\{u, v\}$ form an induced matching of G , which is a contradiction.

Step 2. If n has 3 prime factors then $n = pqr$. Assume by contradiction that $n = p^a q^b r^c$ for some $a > 1$ and b, c are positive integers. Let $x = p^a$, $y = q^b r^c$, $u = p^{a-1} q^b$, and $v = pr^c$. Then $\{x, y\}$ and $\{u, v\}$ form an induced matching of G , a contradiction.

Step 3. If n has 2 prime factors then $n = p^a q$. Assume by contradiction that $n = p^a q^b$ where $a, b > 1$ are positive integers. Let $x = p^a$, $y = q^b$, $u = p^{a-1} q^{b-1}$ and $v = pqw$ where w is any prime divisor of $pq - 1$. Then $u \neq v$ and $\{x, y\}$ and $\{u, v\}$ form an induced matching of G , a contradiction.

The conclusion then follows from Lemma 3.2, Lemma 3.3, and Lemma 3.4. $\square$

We now apply Theorem 1.7 to compute all the Betti numbers of $\Gamma(\mathbb{Z}_n)$ when n is of the form $n = p^a$, $n = p^a q$, or $n = pqr$.

Proof of Theorem 1.8. Let $\mathbf{s}_j = (s_{j,1}, \dots, s_{j,\varphi_j})$ with $\varphi_j = p^{a-j-1}(p-1)$ and

$$\begin{aligned} s_{j,1} &= p^j - p^{a-j-1} - 1 \\ s_{j,\ell} &= s_{j,\ell-1} - 1 \text{ for } \ell = 2, \dots, \varphi_j. \end{aligned}$$

By Lemma 3.2, $\Gamma(\mathbb{Z}_n)$ has a type $\mathbf{s} = (\mathbf{s}_{a-1}, \dots, \mathbf{s}_{\lceil \frac{a}{2} \rceil})$. By Theorem 1.7, we deduce that

$$\beta_i(S/I) = \sum_{j=\lceil \frac{a}{2} \rceil}^{a-1} \varphi_j \left(s_{j,1} + \sum_{i=\lceil \frac{a}{2} \rceil}^{a-1} \varphi_i \right).$$

For $j < a - 1$, we have

$$(3.1) \quad \sum_{\ell=j+1}^{a-1} \varphi_\ell = (p-1) \sum_{\ell=j+1}^{a-1} p^{a-\ell-1} = p^{a-j-1} - 1.$$

The conclusion follows. $\square$

Proof of Theorem 1.9. Let $\mathbf{s}_j = (s_{j,1}, \dots, s_{j,\varphi_j})$ with $\varphi_j = p^{a-j-1}(p-1)$ be such that $s_{j,\ell} = s_{j,\ell-1} - 1$ for $\ell = 2, \dots, \varphi_j$ and

$$\begin{aligned} s_{j,1} &= qp^j - p^{a-j-1} - 1 \text{ for } j = \lceil \frac{a}{2} \rceil, \dots, a-1 \\ s_{j,1} &= (q-1)p^j + p^{a-j-1}(p-1) - 1 \text{ for } j = 0, \dots, \lceil \frac{a}{2} \rceil - 1. \end{aligned}$$

By Lemma 3.3, Theorem 1.7, and the proof of Theorem 1.8, we deduce that

$$\beta_i(S/I) = \sum_{j=0}^{a-1} \varphi_j \left(s_{j,1} + \sum_i^{a-1} \varphi_\ell \right).$$

The conclusion then follows from Eq. (3.1). $\square$

Proof of Theorem 1.10. The conclusion follows from Lemma 3.4 and Theorem 1.7. $\square$

Remark 3.5. Rather, Imran, and Pirzada [RIP] and Pirzada and Rather [PR] used Hochster’s formula [HH, Theorem 8.1.1] to compute the Betti numbers of the edge ideals of Γ_n when n is of the form p^a , $p^a q$, and pqr . The authors provided closed formulae when $a \leq 4$ in the first case and $a \leq 2$ in the second case. Our method is entirely different and yields a complete answer. Additionally, our formulae are simpler. By Theorem 1.10, the last Betti number of Γ_{pqr} is $p - 1$. Hence, the the first statement in [RIP, Proposition 4.8] is incorrect.

Remark 3.6. Theorem 1.7 is useful for calculating the Betti numbers of other classes of cochordal graphs. We will explore this further in subsequent work.

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DEPARTMENT OF ALGEBRA AND GEOMETRY, HONG DUC UNIVERSITY, NO. 565 QUANG TRUNG STREET, DONG VE WARD, THANH HOA, VIETNAM

Email address: lexuandung@hdu.edu.vn

INSTITUTE OF MATHEMATICS, VAST, 18 HOANG QUOC VIET, HANOI, VIETNAM

Email address: vuqthanh@gmail.com