Persistence probabilities for MA(1) sequences with uniform innovations

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Abstract

We study the persistence probabilities of a moving average process of order one with uniform innovations. We identify a number of regions—characterized by the location of the uniform distribution and the coupling parameter of the process—where the persistence probabilities have qualitatively different behaviour. We obtain the generating functions of the persistence probabilities explicitly in all possible regions. In some of the regions, the persistence probabilities can be expressed explicitly in terms of various combinatorial quantities.

Keywords: persistence probability; moving average process; exit time; Mallows-Riordan polynomials

1 Introduction and main results

In recent years, there has been a lot of interest in the study of the persistence probabilities of various stochastic processes, many of them with relations to different physical systems. The persistence probability related to a one-dimensional stochastic process is simply the probability that the process does not change sign in a finite number of steps. In physical terms, the persistence probabilities (and in particular their the rate of decay) measure how fast an underlying physical system started in a non-equilibrium state returns to equilibrium. We refer to the survey papers [8, 20] and the monograph [17] for an overview on the relevance of the question to physical systems and to [5] for a survey of the mathematical literature.

In the present paper, we aim to compute the persistence probabilities of a moving average process of order one with uniform innovations. More concretely, for $n \geqslant 1$ and any coupling parameter $\theta \in \mathbb{R}$, let us look at the probability

$$p_n^a(\theta) := \mathbb{P}(X_2 \geqslant \theta X_1, X_3 \geqslant \theta X_2, \ldots, X_{n+1} \geqslant \theta X_n), \tag{1}$$

when the (X_i) are i.i.d. uniform on $[-a, 1]$ with either $a \geqslant 0$ or $a \in (-1, 0]$ (with the convenient abbreviation $p_0^a(\theta) := 1$). Let us mention that fixing the right end-point of the uniform distribution at 1 is w.l.o.g., so that we treat all possible uniform distributions. Fixing the right end point is for notational simplicity.

We note that (1) is the persistence (i.e. non-negativity) probability for the MA(1) process $Z_n := X_n - \theta X_{n-1}$. The present paper continues the study initiated in [15] where the symmetric case $a = 1$ is treated.

Another way of looking at the problem is as follows: Consider the $n+1$-dimensional cube $[-a, 1]^{n+1}$. We cut this cube along the hyperplanes $\theta x_i = x_{i+1}$, $i = 1, \ldots, n$, and keep only the part with $\theta x_i < x_{i+1}$ for all i. The question we treat here is: How large the volume of the remaining polytope (relative to the volume $(1 + a)^{n+1}$), depending on a and θ?

Before we can state our main result, one more notation is needed: Define for $z \in \mathbb{C}$ and $r \in \mathbb{C}$ with $|r| < 1$ the deformed exponential function by

$$E(r, z) := \sum_{n=0}^{\infty} \frac{r^{n(n-1)/2}}{n!} z^n.$$

Given this notation, we can already present our main results. Depending on the parameters $a \in (-1, \infty)$ and $\theta \in \mathbb{R}$, the persistence probabilities have a qualitatively different form. The different regions of the parameters with similar structure are depicted in Figure 1. We now summarize the results. Their proofs and further refinements and corollaries can be found in subsequent subsections.

The results give the explicit generating function of the persistence probabilities $(p_n^a(\theta))$ in the different regions.

Theorem 1. (B) *Let either $\theta \in [0, 1]$ and $a \geqslant 0$ or $\theta \in [-1, 0]$ and $a \in [-\theta, -\frac{1}{\theta}]$. Then*

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \frac{E\left(\theta, \frac{az}{1+a}\right) - E\left(\theta, \frac{-z}{1+a}\right)}{zE\left(\theta, \frac{-z}{1+a}\right)}. \tag{2}$$

(G) *Let $\theta \in [-1, 0]$ and $a \geqslant -1/\theta$. Then*

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = 1 + \frac{1 - \theta}{\theta(1 + a)} + \frac{E(\theta, \frac{-z}{\theta(1+a)}) - E(\theta, \frac{-z}{1+a})}{zE(\theta, \frac{-z}{1+a})}. \tag{3}$$

(Y) *Let $\theta \in [-1, 0]$ and $a \in [0, -\theta]$. Then*

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \frac{1}{z} \left(\frac{1}{\frac{E(\theta, \frac{az}{\theta(1+a)})}{E(\theta, \frac{az}{1+a})} - \frac{1 + \frac{a}{\theta}}{1+a} z} - 1 \right). \tag{4}$$

(O) *Let $\theta \in (0,1]$ and $-\theta \leqslant a \leqslant 0$. Then, with $x_+ := \min(x,0)$,*

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \frac{\sum_{i=0}^{\infty}(-1)^i \frac{(\theta^i+a)_+^{i+1}}{(i+1)!\theta^{i(i+1)/2}(1+a)^{i+1}} z^i}{1 - z \cdot \sum_{i=0}^{\infty}(-1)^i \frac{(\theta^i+a)_+^{i+1}}{(i+1)!\theta^{i(i+1)/2}(1+a)^{i+1}} z^i}. \tag{5}$$

(W) *Let $\theta \in [-1,1]$ and $-1 < a \leqslant (-\theta) \wedge 0$. Then $p_n^a(\theta) = 1$ for all $n \geqslant 0$.*
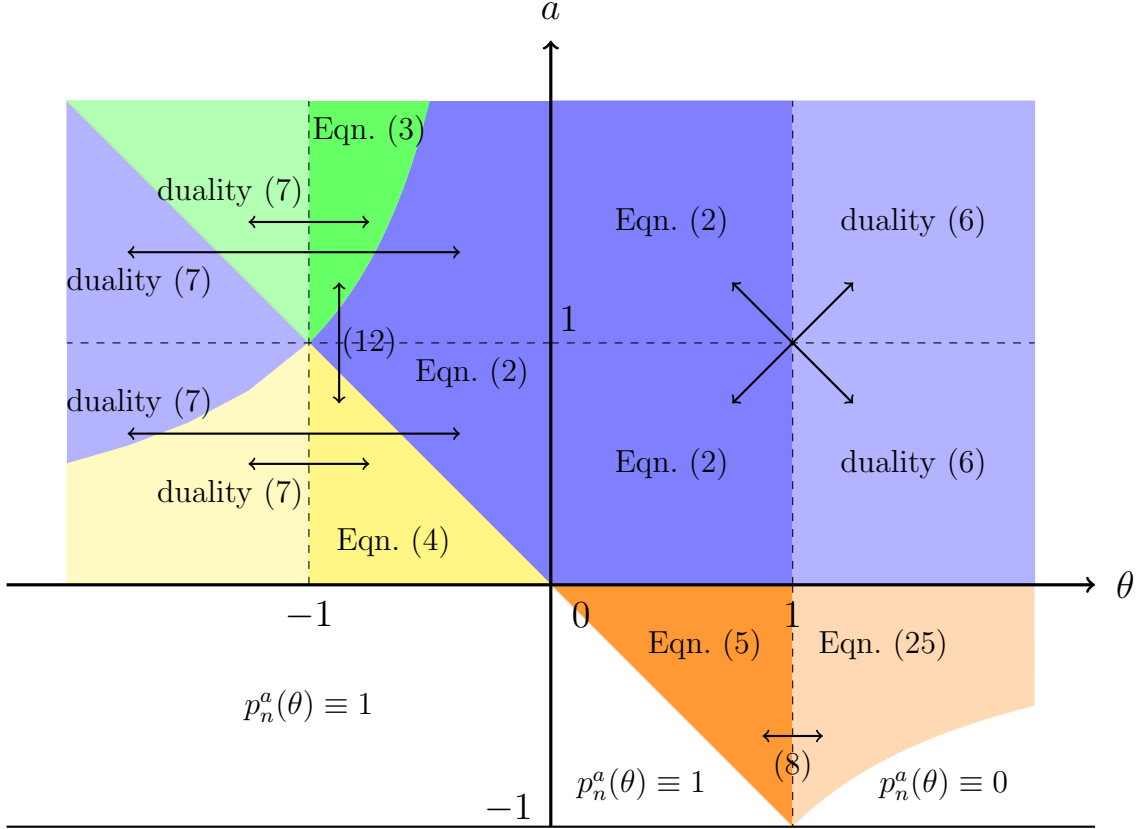

Figure 1: Phase diagram of the main results

It is worth noting that in the study of persistence problems, explicit and comprehensive results like Theorem 1 are rather rare; most works instead concentrate on asymptotic behavior, often through the persistence exponent.

A few comments on these results are due:

1. We marked the different regions with colours for a better overview. Correspondingly, the letters in Theorem 1 stand for blue (B), green (G), yellow (Y), orange (O), and white (W), cf. Figure 1.

2. The theorem only considers the case $\theta \in [-1,1]$. The persistence probabilities for $|\theta| > 1$ can be obtained immediately from Theorem 1 and duality formulas given

in Lemma 1 (treating all $a \geqslant 0$) and Lemma 2 (treating in particular $a \in (-1, 0]$). The dualities are of interest in their own right.

3. It is reasonable to ask for which z the statements in Theorem 1 hold. This is to be understood as follows: Let z_0 be the zero of the denominator of the function on the right-hand side with smallest modulus. Then the quantity on the left-hand side, i.e. the generating function of the $(p_n^a(\theta))$, is finite for all z with $|z| < |z_0|$ and both quantities coincide in this case. Also cf. Remark 7 below. Note that the quantity on the right-hand sides is finite for all $z \in \mathbb{C}$ except for the zeros of the denominators, respectively. Alternatively, the equalities stated in Theorem 1 may also be interpreted as identities between formal power series.

4. The blue, green, and yellow regions are closely interconnected. More precisely, the formula in the blue region can be transferred into the green one through a conditioning argument, and, by a duality formula in Corollary 1, the formula in the green region is equivalent to a formula in the yellow region.

5. For $\theta = 1$ and any $a > -1$, we have $p_n^a(1) = \frac{1}{(n+1)!}$, as can be seen from the borderline of the blue region (B) and the orange region (O). This result is due to [15] and [12] using different proofs. We give a proof here in Lemma 4 below in order to make this paper fully self-contained.

6. We further remark that the result for the symmetric case $a = 1$ was obtained already in [15], see their equations (35) and (36).

7. Of great interest in the study of persistence probabilities is typically the exponential decay rate, also called persistence exponent. In all examples in Theorem 1, the persistence exponent can be obtained as the smallest zero $z_0 =: 1/\lambda$ of the denominator of the generating function. For example, in the blue region, let $z_0 =: 1/\lambda$ be the smallest zero z of the equation $E(\theta, \frac{-z}{1+a}) = 0$. Then it follows from Theorem 1 that
$$p_n^a(\theta) \sim \frac{(1+a)E(\theta, \frac{a}{\lambda(1+a)})}{E(\theta, \frac{-\theta}{\lambda(1+a)})} \cdot \lambda^{n+2}, \qquad \text{as } n \to \infty.$$

8. Note that, in the orange region (O), the generating function of the persistence probabilities (5) is piecewise rational; in the light orange region ($\theta > 1$, $a \in [-1/\theta, 0)$), it is even piecewise polynomial. Contrary, in the blue, green, and yellow region, respectively, the generating function consists of a combination of the (transcendent) deformed exponential functions $E(.,.)$ with different arguments. This type of classification of a probabilistic problem—based on the algebraic nature (rational vs. transcendental) of the generating function—recalls a combinatorial model that has been extensively studied in the literature over the past two decades: walks in the quarter plane; see [7]. In that context, the central question was to classify the transition weights according to whether the counting generating function is rational, algebraic, or transcendental.

9. In the blue region, see (2), and in the green region, see (3), one can express the persistence probabilities $p_n^a(\theta)$ explicitly in terms of purely combinatorial quantities, see Corollaries 5, 6, 7, and 9.

Related work. Let us mention some related papers. We already cited [15] that deals with the symmetric case $a = 1$, which was the starting point of our work. In [4], general Markov chains and their non-exit probabilities from sets are studied. A particular case are moving average processes (with general innovation distribution and general order), for which the persistence exponent is characterized as the leading eigenvalue of an eigenvalue equation. This eigenvalue equation is studied with perturbation techniques in [2] for the standard normal innovation distribution. A further related work is [12], where estimates are given for the Gaussian case and the result of the present paper for the special case $\theta = 1$ appears. See [10] for an analysis of the records for the moving average of a time series, in particular for uniform innovations.

We further mention [1], which treat the persistence probabilities of order one autoregressive processes with uniform innovations using similar techniques. The same question for autoregressive processes is treated in [3, 4, 6, 9, 11, 14, 18, 19, 22] with different methods. Finally, we mention that we deal with zeros of the deformed exponential function in some cases, which are also an interesting object of study in its own right, see e.g. [13, 21, 23] and references therein.

Outline. The structure of this paper is as follows. In Section 2, we formulate and prove four crucial dualities, which in particular allow to compute the $p_n^a(\theta)$ in terms of $p_n^a(1/\theta)$ or $p_n^{1/a}(1/\theta)$ or $p_n^{1/a}(\theta)$, respectively. Section 3 is devoted to certain combinatorial quantities which describe the series representation of $1/E(\theta, z)$ and which are used later to express the $p_n^a(\theta)$ explicitly. Finally, Sections 4, 6, 5, and 7 contain the proofs of Theorem 1 in the blue, green, orange, and yellow region, respectively. Furthermore, a number of additional properties and refinements are given in those sections; in particular, the persistence probabilities can be re-written in terms of the combinatorial quantities from Section 3 or as Mallows-Riordan polynomials in some cases.

2   Dualities and trivial cases

We start with four crucial dualities that allow to reduce some cases to others.

Lemma 1. *Let $a > 0$. Then for $\theta > 0$, we have*

$$p_n^a(\theta) = p_n^{1/a}(1/\theta). \tag{6}$$

For $\theta < 0$, we have

$$p_n^a(\theta) = p_n^a(1/\theta). \tag{7}$$

Proof. Throughout, we will use $\bar{\theta} := 1/\theta$. Let $\theta > 0$. Then

$$
\begin{aligned}
p_n^a(\theta) &= \mathbb{P}(X_2 \geqslant \theta X_1, X_3 \geqslant \theta X_2, \ldots, X_{n+1} \geqslant \theta X_n) \\
&= \mathbb{P}(\bar{\theta} X_2 \geqslant X_1, \bar{\theta} X_3 \geqslant X_2, \ldots, \bar{\theta} X_{n+1} \geqslant X_n) \\
&= \mathbb{P}(\bar{\theta} X_n \geqslant X_{n+1}, \bar{\theta} X_{n-1} \geqslant X_n, \ldots, \bar{\theta} X_1 \geqslant X_2) \\
&= \mathbb{P}(\bar{\theta} X_1 \geqslant X_2, \ldots, \bar{\theta} X_{n-1} \geqslant X_n, \bar{\theta} X_n \geqslant X_{n+1}) \\
&= \mathbb{P}(\bar{\theta}\,(-X_1/a) \leqslant -X_2/a, \ldots, \bar{\theta}\,(-X_{n-1}/a) \leqslant -X_n/a, \bar{\theta}\,(-X_n/a) \leqslant -X_{n+1}/a) \\
&= p_n^{1/a}(1/\theta).
\end{aligned}
$$

In the last computation, we used that $\theta > 0$ (second step), the fact that $(X_1, \ldots, X_{n+1}) \overset{d}{=} (X_{n+1}, \ldots, X_1)$ (third step), the assumption that $a > 0$ (fifth step), and the fact that the $-X_i/a$ are uniformly distributed on $[-1/a, 1]$ (last step).

Let $\theta < 0$. Then, with a similar, but even simpler computation, we obtain:

$$
\begin{aligned}
p_n^a(\theta) &= \mathbb{P}(X_2 \geqslant \theta X_1, X_3 \geqslant \theta X_2, \ldots, X_{n+1} \geqslant \theta X_n) \\
&= \mathbb{P}(\bar{\theta} X_2 \leqslant X_1, \bar{\theta} X_3 \leqslant X_2, \ldots, \bar{\theta} X_{n+1} \leqslant X_n) \\
&= \mathbb{P}(\bar{\theta} X_n \leqslant X_{n+1}, \bar{\theta} X_{n-1} \leqslant X_n, \ldots, \bar{\theta} X_1 \leqslant X_2) \\
&= \mathbb{P}(\bar{\theta} X_1 \leqslant X_2, \ldots, \bar{\theta} X_{n-1} \leqslant X_n, \bar{\theta} X_n \leqslant X_{n+1}) \\
&= p_n^a(1/\theta).
\end{aligned}
$$

We used that $\theta < 0$ (second step) and the fact that $(X_1, \ldots, X_{n+1}) \overset{d}{=} (X_{n+1}, \ldots, X_1)$ (third step). $\qquad\square$

We stress that the next duality is valid for general continuous distributions.

Lemma 2. *Let (X_i) be i.i.d. random variables with continuous distribution and $\theta > 0$. Set $p_0(\theta) := 1$ and, for $n \geqslant 1$, $p_n(\theta)$ as in (1). Then*

$$
p_n(\theta) = \sum_{k=1}^{n} p_{n-k}(\theta) \cdot p_{k-1}(1/\theta)(-1)^{k-1} + (-1)^n p_n(1/\theta), \qquad n \geqslant 0. \tag{8}
$$

Furthermore, denoting $\hat{P}_\theta(z) := \sum_{n=0}^{\infty} p_n(\theta) z^n$, we have

$$
\hat{P}_\theta(z) = \frac{\hat{P}_{1/\theta}(-z)}{1 - z\hat{P}_{1/\theta}(-z)}. \tag{9}
$$

Proof. Let $\bar{\theta} := 1/\theta$. Set $A_i := \{\theta X_i \leqslant X_{i+1}\}$ and $B_i := \{X_i \geqslant \bar{\theta} X_{i+1}\}$. Notice that B_i is the complement of A_i up to a set of probability zero, because the X_i have a continuous distribution. Therefore,

$$
\begin{aligned}
p_n(\theta) &= \mathbb{P}(A_1 \cap \ldots \cap A_n) \\
&= \mathbb{P}(A_1 \cap \ldots \cap A_{n-1}) - \mathbb{P}(A_1 \cap \ldots \cap A_{n-1} \cap B_n) \\
&= \mathbb{P}(A_1 \cap \ldots \cap A_{n-1}) - (\mathbb{P}(A_1 \cap \ldots \cap A_{n-2} \cap B_n) - \mathbb{P}(A_1 \cap \ldots \cap A_{n-2} \cap B_{n-1} \cap B_n))
\end{aligned}
$$

$$= \mathbb{P}(A_1 \cap \ldots \cap A_{n-1}) - \mathbb{P}(A_1 \cap \ldots \cap A_{n-2}) \cdot \mathbb{P}(B_n) + \mathbb{P}(A_1 \cap \ldots \cap A_{n-2} \cap B_{n-1} \cap B_n)$$
$$= p_{n-1}(\theta) - p_{n-2}(\theta) \cdot p_1(\bar{\theta}) + \mathbb{P}(A_1 \cap \ldots \cap A_{n-3} \cap B_{n-1} \cap B_n)$$
$$- \mathbb{P}(A_1 \cap \ldots \cap A_{n-3} \cap B_{n-2} \cap B_{n-1} \cap B_n),$$

because $A_1 \cap \ldots \cap A_{n-2}$ and B_n are independent and, by exchangeability,

$$\mathbb{P}(B_i \cap \ldots \cap B_n) = \mathbb{P}(X_i \geqslant \bar{\theta}X_{i+1}, \ldots, X_n \geqslant \bar{\theta}X_{n+1}) \tag{10}$$
$$= \mathbb{P}(X_{n-i+2} \geqslant \bar{\theta}X_{n-i+1}, \ldots, X_2 \geqslant \bar{\theta}X_1) = p_{n-i+1}(\bar{\theta}).$$

Continuing this way, we obtain that

$$p_n(\theta) = \sum_{k=1}^{n} (-1)^{k-1} p_{n-k}(\theta) \cdot p_{k-1}(\bar{\theta}) + (-1)^n p_n(\bar{\theta}),$$

which is (8). Multiplying by z^n and summing in n, we obtain

$$\hat{P}_\theta(z) - 1 = \sum_{n=1}^{\infty} p_n(\theta) z^n$$
$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} p_{n-k}(\theta) z^{n-k} \cdot z p_{k-1}(\bar{\theta})(-1)^{k-1} z^{k-1} + \sum_{n=1}^{\infty} (-1)^n p_n(\bar{\theta}) z^n$$
$$= \hat{P}_\theta(z) \cdot z \hat{P}_{\bar{\theta}}(-z) + \hat{P}_{\bar{\theta}}(-z) - 1,$$

as claimed in (9). $\qquad\square$

Also the next lemma is valid for general continuous distributions.

Lemma 3. *Let (X_i) be i.i.d. random variables with continuous distribution and $\theta < 0$. Set $p_0(\theta) := 1$ and, for $n \geqslant 1$, $p_n(\theta)$ as in (1), and*

$$q_n(\theta) := \mathbb{P}(\theta X_1 \geqslant X_2, \ldots, \theta X_n \geqslant X_{n+1}).$$

Then

$$p_n(\theta) = \sum_{k=1}^{n} p_{n-k}(\theta) \cdot q_{k-1}(1/\theta)(-1)^{k-1} + (-1)^n q_n(1/\theta), \qquad n \geqslant 0.$$

Furthermore, denoting $\hat{P}_\theta(z) := \sum_{n=0}^{\infty} p_n(\theta) z^n$ and $\hat{Q}_\theta(z) := \sum_{n=0}^{\infty} q_n(\theta) z^n$, we have

$$\hat{P}_\theta(z) = \frac{\hat{Q}_{1/\theta}(-z)}{1 - z\hat{Q}_{1/\theta}(-z)}. \tag{11}$$

Proof. We proceed exactly as in the proof of Lemma 8. Set $\bar{\theta} := 1/\theta$. This time, since $\theta < 0$, $A_i := \{\theta X_i \leqslant X_{i+1}\} = \{X_i \geqslant \bar{\theta}X_{i+1}\}$, $B_i := \{X_i \leqslant \bar{\theta}X_{i+1}\}$. By the assumption that the X_i have a continuous distribution, B_i is the complement of A_i up to a set of probability zero. We can now follow line-by-line the proof of Lemma 8. The only difference is that this time, instead of (10), we have

$$\mathbb{P}(B_i \cap \ldots \cap B_n) = \mathbb{P}(X_i \leqslant \bar{\theta}X_{i+1}, \ldots, X_n \leqslant \bar{\theta}X_{n+1})$$
$$= \mathbb{P}(X_{n-i+2} \leqslant \bar{\theta}X_{n-i+1}, \ldots, X_2 \leqslant \bar{\theta}X_1) = q_{n-i+1}(\bar{\theta}),$$

thereby completing the proof. $\qquad\square$

Let us specify the last result for uniform distributions.

Corollary 1. *Let $\theta < 0$ and $a > 0$. Set $\hat{P}_{a,\theta}(z) := \sum_{n=0}^{\infty} p_n^a(\theta) z^n$. Then*

$$\hat{P}_{a,\theta}(z) = \frac{\hat{P}_{1/a,\theta}(-z)}{1 - z\hat{P}_{1/a,\theta}(-z)}. \tag{12}$$

Proof. We use Lemma 3 and observe that

$$\begin{aligned}
q_n(\theta) &= \mathbb{P}(\bar{\theta}X_1 \geqslant X_2, \ldots, \bar{\theta}X_n \geqslant X_{n+1}) \\
&= \mathbb{P}(\bar{\theta}(-X_1/a) \leqslant (-X_2/a), \ldots, \bar{\theta}(-X_n/a) \geqslant (-X_{n+1}/a)) \\
&= p_n^{1/a}(\bar{\theta}) = p_n^{1/a}(\theta),
\end{aligned}$$

where we used that the $-X_i/a$ are uniform on $[-1/a, 1]$ (in the third step) and the duality (7) in the last step. Therefore, the generating function on the right-hand side in (11) becomes $\hat{P}_{1/a,\theta}$, while the generating function on the left-hand side in (11) is $\hat{P}_{a,\theta}$, as claimed. $\square$

After treating the crucial dualities, we mention a few trivial cases. Note that Lemma 4(a)(i) proves the result claimed in the white case of Theorem 1.

Lemma 4. (a) *Let $a \in (-1, 0]$.*

(i) *For $\theta \leqslant -a$ we have $p_n^a(\theta) = 1$ for all $n \geqslant 0$.*

(ii) *For $\theta \geqslant -1/a$ we have $p_n^a(\theta) = 0$ for all $n \geqslant 1$.*

(b) *For any $a \in (-1, \infty)$ and $\theta = 1$ we have*

$$p_n^a(1) = \frac{1}{(n+1)!}.$$

Let us stress that the result in (b) for $\theta = 1$ (and in fact also its proof) is valid for general continuous distributions. It was first proved by [15] (see their (28)) and re-proved in [12] (see their Example 6). We mention the proof here for completeness.

Let us further remark that the statements in Lemma 4(a)(i) and (a)(ii) are equivalent by the duality (8).

Proof. First, we prove part (i) of (a). Let $a \in (-1, 0]$ and $0 \leqslant \theta \leqslant -a$. Then for any i

$$\theta X_i \leqslant \theta \leqslant -a \leqslant X_{i+1},$$

so that $p_n^a(\theta) = 1$. On the other hand, let $a \in (-1, 0]$ and $\theta \leqslant 0$. Then for any i

$$\theta X_i \leqslant 0 \leqslant -a \leqslant X_{i+1},$$

so that $p_n^a(\theta) = 1$.

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Next, we prove part (ii) of (a). If $a \in (-1, 0)$ and $\theta \geqslant -1/a$ we have for any i

$$\theta X_i \geqslant \frac{-1}{a} X_i \geqslant \frac{-1}{a} \cdot (-a) = 1 \geqslant X_{i+1},$$

so that $p_n^a(\theta) = 0$.

Let us finally show part (b). Note that for any permutation π of the set $\{1, \ldots, n+1\}$, by exchangeability,

$$\mathbb{P}(X_{\pi(1)} \leqslant X_{\pi(2)} \leqslant \ldots \leqslant X_{\pi(n)} \leqslant X_{\pi(n+1)}) = \mathbb{P}(X_1 \leqslant X_2 \leqslant \ldots \leqslant X_n \leqslant X_{n+1}) = p_n^a(1).$$

Summing this over all π and simplifying (and using the fact that the distribution of (X_i) is continuous) immediately implies the claim. $\qquad\square$

3   A combinatorial lemma

In some of the cases that we treat, expressions of the type $\frac{1}{E(\theta, z)}$ appear, where E is the deformed exponential function, cf. (2), (4), and (3). The main lemma of this section, Lemma 5, which may be of independent interest, gives a purely combinatorial description of the terms in the series expansion of $\frac{1}{E(\theta, z)}$ at $z = 0$.

For this purpose, we need a notion to describe certain combinatorial quantities.

Definition 1. *Fix an integer $\ell \geqslant 1$. An ℓ-profile is a vector $\mathbf{r} = (r_1, \ldots, r_\ell)$ with $r_i \geqslant 0$ and $\sum_{i=1}^{\ell} i r_i = \ell$.*

Let us mention a few examples: The only 1-profile is $\mathbf{r} = (1)$. The only two 2-profiles are $\mathbf{r} = (2, 0)$ and $\mathbf{r} = (0, 1)$. The only three 3-profiles are $\mathbf{r} = (3, 0, 0)$, $\mathbf{r} = (1, 1, 0)$, and $\mathbf{r} = (0, 0, 1)$.

Definition 2. *Fix $k \geqslant 1$ and $\ell \geqslant 1$. For an integer vector $\mathbf{n} = (n_1, \ldots, n_k)$ with $n_i \geqslant 0$ and $n_1 + \ldots + n_k = \ell$, we define the ℓ-profile of $\mathbf{n}$ by*

$$r(\mathbf{n}) := (r_1(\mathbf{n}), \ldots, r_\ell(\mathbf{n})), \qquad \textit{where} \qquad r_i(\mathbf{n}) := |\{j \mid n_j = i\}| = \sum_{j=1}^{k} \mathbb{1}_{n_j = i}.$$

Note that $k - \sum_{i=1}^{\ell} r_i(\mathbf{n})$ is the number of zeros in the integer vector $(n_1, \ldots, n_k)$. For an ℓ-profile $\mathbf{r}$, set

$$w_k(\mathbf{r}) := \sum_{n_1 + \ldots + n_k = \ell, n_i \geqslant 0, r(\mathbf{n}) = \mathbf{r}} 1.$$

This is the number of distinct ways to obtain the ℓ-profile $\mathbf{r}$ by using integer vectors $\mathbf{n}$ of length k. A simple combinatorial fact is that for any ℓ-profile $\mathbf{r} = (r_1, \ldots, r_\ell)$,

$$w_k(\mathbf{r}) = \binom{k}{r_1, \ldots, r_\ell, k - \sum_{i=1}^{\ell} r_i} = \frac{k!}{(\prod_{j=1}^{\ell} r_j!) \cdot (k - \sum_{i=1}^{\ell} r_i)!}. \qquad (13)$$

We can now state and prove the main lemma of this section.

Lemma 5. *Let $\theta \in \mathbb{C}$ with $|\theta| < 1$ and $z \in \mathbb{C}$. Then*

$$\frac{1}{E(\theta, z)} = \sum_{\ell=0}^{\infty} \phi_\ell(\theta) z^\ell,$$

where $\phi_0(\theta) := 1$ and

$$\phi_\ell(\theta) := \sum_{\ell\text{-profile } \mathbf{r}} \binom{\sum_{i=1}^{\ell} r_i}{r_1, \ldots, r_\ell} \prod_{i=1}^{\ell} \left[\frac{-\theta^{i(i-1)/2}}{i!} \right]^{r_i}. \tag{14}$$

The statement of the lemma has to be understood in the sense of item number 3 in the remarks following Theorem 1.

Proof. In a first step, we use the geometric series and write $1/E(\theta, z)$ as a series in z with coefficients that consist of multiple integrals:

$$\frac{1}{E(\theta, z)} = \frac{1}{1 - \sum_{n=1}^{\infty} \frac{-\theta^{\frac{n(n-1)}{2}}}{n!} z^n}$$

$$= 1 + \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{-\theta^{\frac{n(n-1)}{2}}}{n!} z^n \right)^k$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1!} \cdots \frac{1}{n_k!} z^{n_1 + \cdots + n_k} \theta^{\frac{n_1(n_1-1)}{2} + \cdots + \frac{n_k(n_k-1)}{2}}$$

$$= 1 + \sum_{\ell=1}^{\infty} z^\ell \sum_{k=1}^{\infty} (-1)^k \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \mathbb{1}_{n_1 + \cdots + n_k = \ell} \frac{1}{n_1! \ldots n_k!} \theta^{\frac{n_1(n_1-1)}{2} + \cdots + \frac{n_k(n_k-1)}{2}}$$

$$= 1 + \sum_{\ell=1}^{\infty} z^\ell \sum_{k=1}^{\ell} (-1)^k \sum_{n'_1=1}^{\ell} \cdots \sum_{n'_k=1}^{\ell} \mathbb{1}_{n'_1 + \cdots + n'_k = \ell} \frac{1}{n'_1! \ldots n'_k!} \theta^{\frac{n'_1(n'_1-1)}{2} + \cdots + \frac{n'_k(n'_k-1)}{2}}. \tag{15}$$

We shall continue to compute the coefficients of the series (only). We use the notation $\kappa(\mathbf{n})$ for the number of zeros in the vector $\mathbf{n} = (n_1, \ldots, n_\ell)$, i.e. $\kappa(\mathbf{n}) := \sum_{i=1}^{\ell} \mathbb{1}_{n_i=0}$. Recall that $\kappa(\mathbf{n}) = \ell - \sum_{i=1}^{\ell} r_i(\mathbf{n})$. In the following computation, the combinatorial factor $\binom{\ell}{\ell-k(\mathbf{n})}$ enters in the second step because one can choose the location of the $\ell - k(\mathbf{n}) = k$ non-zeros in the vector $(n_1, \ldots, n_\ell)$ with $n_1 + \ldots + n_\ell = \ell$ and $n_i \geqslant 0$. Having chosen those gives the vector $(n'_1, \ldots, n'_k)$ with $n'_i \geqslant 1$. This allows us to compute the coefficients:

$$\sum_{k=1}^{\ell} (-1)^k \sum_{n'_1=1}^{\ell} \cdots \sum_{n'_k=1}^{\ell} \mathbb{1}_{n'_1 + \ldots + n'_k = \ell} \prod_{j=1}^{k} \frac{\theta^{n'_j(n'_j-1)2/2}}{n'_j!}$$

$$= \sum_{n_1 + \ldots + n_\ell = \ell, n_i \geqslant 0} \frac{1}{\binom{\ell}{\ell-\kappa(\mathbf{n})}} (-1)^{\ell-\kappa(\mathbf{n})} \prod_{j=1}^{\ell} \frac{\theta^{n_j(n_j-1)2/2}}{n_j!}$$

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$$= \sum_{\ell\text{-profile } \mathbf{r}} \sum_{n_1+\ldots+n_\ell=\ell,n_i\geqslant 0} \mathbb{1}_{r(\mathbf{n})=\mathbf{r}} \frac{1}{\binom{\ell}{\ell-\kappa(\mathbf{n})}}(-1)^{\ell-\kappa(\mathbf{n})} \prod_{j=1}^{\ell} \frac{\theta^{n_j(n_j-1)2/2}}{n_j!}$$

$$= \sum_{\ell\text{-profile } \mathbf{r}} \sum_{n_1+\ldots+n_\ell=\ell,n_i\geqslant 0} \mathbb{1}_{r(\mathbf{n})=\mathbf{r}} \frac{1}{\binom{\ell}{\sum_{i=1}^{\ell} r_i}}(-1)^{\sum_{i=1}^{\ell} r_i} \prod_{i=1}^{\ell} \frac{(\theta^{i(i-1)/2})^{r_i}}{i!^{r_i}}$$

$$= \sum_{\ell\text{-profile } \mathbf{r}} \frac{1}{\binom{\ell}{\ell-\sum_{i=1}^{\ell} r_i}} \prod_{i=1}^{\ell} \frac{(-1)^{r_i}(\theta^{i(i-1)/2})^{r_i}}{i!^{r_i}} \sum_{n_1+\ldots+n_\ell=\ell,n_i\geqslant 0} \mathbb{1}_{r(\mathbf{n})=\mathbf{r}}$$

$$= \sum_{\ell\text{-profile } \mathbf{r}} \frac{1}{\binom{\ell}{\ell-\sum_{i=1}^{\ell} r_i}} \prod_{i=1}^{\ell} \frac{(-1)^{r_i}(\theta^{i(i-1)/2})^{r_i}}{i!^{r_i}} \cdot w_\ell(\mathbf{r})$$

$$= \sum_{\ell\text{-profile } \mathbf{r}} \frac{(\ell-\sum_{i=1}^{\ell} r_i)!(\sum_{i=1}^{\ell} r_i)!}{\ell!} \prod_{i=1}^{\ell} \frac{(-1)^{r_i}(\theta^{i(i-1)/2})^{r_i}}{i!^{r_i}} \cdot \frac{\ell!}{(\prod_{j=1}^{\ell} r_j!)\cdot(\ell-\sum_{i=1}^{\ell} r_i)!}$$

$$= \sum_{\ell\text{-profile } \mathbf{r}} \binom{\sum_{i=1}^{\ell} r_i}{r_1,\ldots,r_\ell} \prod_{i=1}^{\ell} \left[\frac{-\theta^{i(i-1)/2}}{i!}\right]^{r_i},$$

where we used (13) in the sixth step. $\qquad\square$

It is also possible to characterize the number of monomials of $\phi_\ell(\theta)$ in (14). To that purpose, we need to introduce the following notation. Given $\ell \geqslant 1$, one can associate partitions of ℓ as families of integers $n_1 \geqslant \cdots \geqslant n_k \geqslant 0$ such that $n_1 + \cdots + n_k = \ell$. For any given ℓ, we define c_ℓ as the number of different values attained by the sum $n_1^2 + \cdots + n_k^2$, when $n_1 \geqslant \cdots \geqslant n_k \geqslant 0$ run over all partitions of ℓ. This sequence $(c_\ell)_{\ell\geqslant 0}$ starts with $1, 1, 2, 3, 5, 7, 9, 13, 18, 21$; see OEIS sequence A069999 for more information.

Corollary 2. *The number of monomials in the polynomial $\phi_\ell(\theta)$ in (14) is given by c_ℓ, the ℓ-th term in the OEIS sequence A069999.*

Proof. This is a consequence of (15). There are as many monomial terms in $\phi_\ell(\theta)$ as different values of the sums $\frac{n_1^2+\cdots+n_k^2}{2}$, with $(n_1,\ldots,n_k)$ being a partition of ℓ. $\qquad\square$

4   The blue region

In this section, we deal with the case $\theta \in [0,1]$ and $a \geqslant 0$ or $\theta \in [-1,0]$ and $a \in [-\theta,-\frac{1}{\theta}]$.

4.1   Generating functions

We start with a formula for the generating function of the $(p_n^a(\theta))$.

Lemma 6 (Equation (2) of Theorem 1). *Let either $\theta \in [0,1]$ and $a \geqslant 0$ or $\theta \in [-1,0]$ and $a \in [-\theta, -\frac{1}{\theta}]$. Then*

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \frac{E\left(\theta, \frac{az}{1+a}\right) - E\left(\theta, \frac{-z}{1+a}\right)}{zE\left(\theta, \frac{-z}{1+a}\right)}. \tag{16}$$

Proof. Let us start from the integral expression of the persistence probability:

$$p_n^a(\theta) = \frac{1}{(1+a)^{n+1}} \int_{-a}^{1} \int_{\theta x_1}^{1} \cdots \int_{\theta x_n}^{1} 1 \mathrm{d}x_{n+1} \ldots \mathrm{d}x_2 \mathrm{d}x_1.$$

This representation holds because for any $x \in [-a, 1]$ we have $\theta x \in [-a, 1]$, due to the assumptions that either $\theta \in [0,1]$ and $a \geqslant 0$ or $\theta \in [-1,0]$ and $a \in [-\theta, -\frac{1}{\theta}]$.

We use the notation $H_k(x) := \frac{x^k}{k!}$ and recall that $H'_{k+1} = H_k$. We have

$$p_n^a(\theta) = \frac{1}{(1+a)^{n+1}} \int_{-a}^{1} \int_{\theta x_1}^{1} \cdots \int_{\theta x_{n-1}}^{1} (1 - \theta x_n) \mathrm{d}x_n \ldots \mathrm{d}x_2 \mathrm{d}x_1$$

$$= \frac{1}{(1+a)} p_{n-1}^a(\theta) - \frac{1}{(1+a)^{n+1}} \theta \int_{-a}^{1} \int_{\theta x_1}^{1} \cdots \int_{\theta x_{n-1}}^{1} H_1(x_n) \mathrm{d}x_n \ldots \mathrm{d}x_2 \mathrm{d}x_1$$

$$= \frac{1}{(1+a)} p_{n-1}^a(\theta) - \theta H_2(1) \frac{1}{(1+a)^2} p_{n-2}^a(\theta)$$

$$\quad + \frac{1}{(1+a)^{n+1}} \theta^{1+2} \int_{-a}^{1} \int_{\theta x_1}^{1} \cdots \int_{\theta x_{n-2}}^{1} H_2(x_{n-1}) \mathrm{d}x_{n-1} \ldots \mathrm{d}x_2 \mathrm{d}x_1$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \theta^{\frac{k(k-1)}{2}} \frac{1}{(1+a)^k} p_{n-k}^a(\theta) + \frac{1}{(1+a)^{n+1}} (-1)^n \theta^{\frac{n(n+1)}{2}} \int_{-a}^{1} H_n(x_1) \mathrm{d}x_1$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \theta^{\frac{k(k-1)}{2}} \frac{1}{(1+a)^k} p_{n-k}^a(\theta) + \frac{1}{(1+a)^{n+1}} (-1)^n \theta^{\frac{n(n+1)}{2}} \frac{(1 - (-a)^{n+1})}{(n+1)!},$$

with $p_0^a(\theta) = \frac{1+a}{(1+a)^1} = 1$. Let us abbreviate $\hat{P}(z) := \sum_{n=0}^{\infty} p_n^a(\theta) z^n$. Taking generating functions of the last equality, we deduce that

$$\hat{P}(z) - 1 = \sum_{n=0}^{\infty} p_n^a(\theta) z^n - 1$$

$$= \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \theta^{\frac{k(k-1)}{2}} \frac{1}{(1+a)^k} p_{n-k}^a(\theta)$$

$$\quad + \sum_{n=1}^{\infty} z^n \frac{1}{(1+a)^{n+1}} (-1)^n \theta^{\frac{n(n+1)}{2}} \frac{(1 - (-a)^{n+1})}{(n+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \theta^{\frac{k(k-1)}{2}} \frac{z^k}{(1+a)^k} \sum_{n=k}^{\infty} z^{n-k} p_{n-k}^a(\theta)$$

$$\quad + \sum_{n=1}^{\infty} \frac{(-z)^n}{(1+a)^{n+1}} \theta^{\frac{n(n+1)}{2}} \frac{1}{(n+1)!} + \sum_{n=1}^{\infty} \frac{z^n}{(1+a)^{n+1}} \theta^{\frac{n(n+1)}{2}} \frac{a^{n+1}}{(n+1)!}$$

$$= \left(1 - E\left(\theta, \frac{-z}{1+a}\right)\right) \cdot \hat{P}(z) + \sum_{n=2}^{\infty} \frac{(-z)^{n-1}}{(1+a)^n} \theta^{\frac{n(n-1)}{2}} \frac{1}{n!} + \sum_{n=2}^{\infty} \frac{z^{n-1}}{(1+a)^n} \theta^{\frac{n(n-1)}{2}} \frac{a^n}{n!}$$

$$= \left(1 - E\left(\theta, \frac{-z}{1+a}\right)\right) \cdot \hat{P}(z)$$

$$- z^{-1} \sum_{n=2}^{\infty} \frac{(-z)^n}{(1+a)^n} \frac{\theta^{\frac{n(n-1)}{2}}}{n!} + z^{-1} \sum_{n=2}^{\infty} \frac{(az)^n}{(1+a)^n} \frac{\theta^{\frac{n(n-1)}{2}}}{n!}$$

$$= \left(1 - E\left(\theta, \frac{-z}{1+a}\right)\right) \cdot \hat{P}(z)$$

$$- z^{-1} \left[E\left(\theta, \frac{-z}{1+a}\right) + \frac{z}{1+a} - E\left(\theta, \frac{az}{1+a}\right) + \frac{az}{1+a} \right]$$

$$= \left(1 - E\left(\theta, \frac{-z}{1+a}\right)\right) \cdot \hat{P}(z) - z^{-1} \left[E\left(\theta, \frac{-z}{1+a}\right) - E\left(\theta, \frac{az}{1+a}\right) \right] - 1.$$

This gives (16). □

As already pointed out in item 7 in the series of remarks following Theorem 1, the determination of the persistence exponent, that is, the exponential decay rate of $(p_n(\theta))$, follows from Lemma 6. It is directly related to the first zero of the exponential function studied in [13, 21, 23].

Corollary 3. *Let $\theta \in [0,1]$ and $a \geqslant 0$ or $\theta \in [-1,0]$ and $a \in [-\theta, -\frac{1}{\theta}]$. The persistence exponent λ is given by the inverse $\lambda = 1/z$ of the smallest positive root z of the equation $E(\theta, \frac{-z}{1+a}) = 0$.*

4.2   Representation in terms of Mallows-Riordan polynomials

The next corollary shows that the persistence probabilities in the special case $a = -\theta$ can be expressed in terms of Mallows-Riordan polynomials [16]. Define the Mallows-Riordan polynomials as the polynomials $J_n(\theta)$, $n \geqslant 1$, by

$$\log \left(\sum_{n=0}^{\infty} \theta^{n(n-1)/2} \frac{z^n}{n!} \right) =: \sum_{n=1}^{\infty} (\theta - 1)^{n-1} J_n(\theta) \frac{z^n}{n!}. \tag{17}$$

Differentiating (17) with respect to z, we obtain

$$\sum_{n=0}^{\infty} \frac{J_{n+1}(\theta)}{n!} z^n = \frac{\sum_{n=0}^{\infty} \frac{\theta^{n(n+1)/2}}{n!} \left(-\frac{z}{1-\theta}\right)^n}{\sum_{n=0}^{\infty} \frac{\theta^{n(n-1)/2}}{n!} \left(-\frac{z}{1-\theta}\right)^n}, \tag{18}$$

cf. formula (20) in [1].

Corollary 4. *For all $n \geqslant 0$ and $\theta \in [0,1]$, $p_n^{-\theta}(\theta) = \frac{J_{n+2}(\theta)}{(n+1)!}$.*

Proof. Using (16), we find

$$\sum_{n=0}^{\infty} p_n^{-\theta}(\theta)z^n = \frac{1}{z}\left(\frac{E(\theta, -\frac{\theta z}{1-\theta})}{E(\theta, -\frac{z}{1-\theta})} - 1\right) = \frac{1}{z}\left(\frac{\sum_{n=0}^{\infty}\frac{\theta^{n(n-1)/2}}{n!}(-\frac{\theta z}{1-\theta})^n}{\sum_{n=0}^{\infty}\frac{\theta^{n(n-1)/2}}{n!}(-\frac{z}{1-\theta})^n} - 1\right). \tag{19}$$

Shifting the sum and comparing coefficients in (18) and (19), we can conclude. $\square$

Let us introduce the process (W_n) by $W_0 := 0$ and $W_n := X_n + \theta W_{n-1}$, where the (X_i) are i.i.d. uniform on $[-a, 1]$. The process (W_n) is referred to as an autoregressive process of order 1 with uniform innovations. It is shown in Proposition 5.8 of [1] that for $\theta \in [-1, \frac{a}{1+a}]$,

$$\mathbb{P}(W_1 \geqslant 0, \ldots, W_{n+1} \geqslant 0) = \frac{1}{(1+a)^{n+1}}\frac{J_{n+2}(\theta)}{(n+1)!}.$$

We do not have an intuitive explanation for the connection between the above formula (applying to autoregressive processes, called AR(1)) and that of Corollary 4 (which holds for MA(1) processes), other than through direct computation.

4.3 Combinatorial representation

The goal of this subsection is to represent the persistence probabilities in terms of the combinatorial quantities from Lemma 5.

Let us start with the case $a = 0$ and $\theta \in [0, 1]$, which is particularly simple.

Corollary 5. *Let $\theta \in [0, 1]$ and $a = 0$. Then we have:*

$$p_{\ell-1}^0(\theta) = (-1)^\ell \phi_\ell(\theta) = (-1)^\ell \sum_{\ell\text{-profile } \mathbf{r}} \binom{\sum_{i=1}^{\ell} r_i}{r_1, \ldots, r_\ell} \prod_{i=1}^{\ell}\left[\left(\frac{-\theta^{i(i-1)/2}}{i!}\right)^{r_i}\right], \qquad \ell \geqslant 1.$$

Proof. The statement follows immediately from Lemma 5 applied to formula (16), which in the case $a = 0$ becomes

$$\sum_{n=0}^{\infty} p_n^0(\theta)z^n = \frac{1}{z}\left(\frac{1}{E(\theta, -z)} - 1\right) = \frac{1}{z}\left(\sum_{n=0}^{\infty}\phi_n(\theta)(-z)^n - 1\right) = \sum_{\ell=0}^{\infty}(-1)^{\ell+1}\phi_{\ell+1}(\theta)z^\ell,$$

so that the proof is complete. $\square$

From Corollary 5, we can confirm that $p_{\ell-1}^0(0) = 1$ for any ℓ (the responsible ℓ-profile is $(\ell, 0, \ldots, 0)$). Subsequently, we can show that $p_{\ell-1}^0(\theta) = 1 - \frac{\ell-1}{2}\theta + \frac{(\ell-2)(\ell-3)}{8}\theta^2 + O(\theta^3)$ for $\theta \to 0$ (for the linear term, the responsible ℓ-profile is $(\ell-2, 1, 0, \ldots, 0)$, for the quadratic term $(\ell-4, 2, 0, \ldots, 0)$). We can also obtain the monomial with the largest exponent, namely $(-1)^{\ell+1}\theta^{\ell(\ell-1)/2}\frac{1}{\ell!}$ (the responsible ℓ-profile is $(0, \ldots, 0, 1)$).

On the other hand, it is not obvious to us why at $\theta = 1$ we get

$$\frac{1}{\ell!} = p_{\ell-1}^0(1) = (-1)^\ell \sum_{\ell\text{-profile } \mathbf{r}} \binom{\sum_{i=1}^{\ell} r_i}{r_1, \ldots, r_\ell} \prod_{i=1}^{\ell}\left[\left(\frac{-1}{i!}\right)^{r_i}\right].$$

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This follows from evaluating (16) and combining with Corollary 5, but it is not clear how to obtain this directly. Nonetheless, it seems to be an interesting and non-trivial formula. Similarly intriguing formulas can be obtained for further derivatives $\lim_{\theta \searrow 1} \frac{\partial^k}{\partial \theta^k} p^a_{\ell-1}(\theta)$.

We can extend the combinatorial representation in Corollary 5 to the whole blue region.

Corollary 6. *Let $\theta \in [0,1]$ and $a \geqslant 0$ or $\theta \in [-1,0]$ and $a \in [-\theta, -1/\theta]$. Then*

$$p^a_n(\theta) = \left(\frac{-1}{1+a}\right)^{n+1} \sum_{k=0}^{n+1} \phi_{n+1-k}(\theta) \frac{\theta^{\frac{k(k-1)}{2}}}{k!} (-a)^k, \tag{20}$$

where the (ϕ_ℓ) are defined in (14).

Proof. We have from (16)

$$\sum_{n=0}^{\infty} p^a_n(\theta) z^n = \frac{E\left(\theta, \frac{az}{1+a}\right) - E\left(\theta, \frac{-z}{1+a}\right)}{zE\left(\theta, \frac{-z}{1+a}\right)}$$

$$= E(\theta, \tfrac{az}{1+a})(1+a)^{-1} \cdot \frac{1+a}{z} \frac{1 - E(\theta, \frac{-z}{1+a})}{E(\theta, \frac{-z}{1+a})} + \left(E(\theta, \tfrac{az}{1+a}) - 1\right) \frac{1}{z}$$

$$= E(\theta, \tfrac{az}{1+a})(1+a)^{-1} \cdot \sum_{n=0}^{\infty} p^0_n(\theta) \left(\tfrac{z}{1+a}\right)^n + \left(E(\theta, \tfrac{az}{1+a}) - 1\right) \frac{1}{z}$$

$$= \frac{1}{1+a} \sum_{k=0}^{\infty} \frac{\theta^{\frac{k(k-1)}{2}}}{k!} \left(\tfrac{az}{1+a}\right)^k \cdot \sum_{n=0}^{\infty} p^0_n(\theta) \left(\tfrac{z}{1+a}\right)^n + \left(\sum_{k=0}^{\infty} \frac{\theta^{\frac{k(k-1)}{2}}}{k!} \left(\tfrac{az}{1+a}\right)^k - 1\right) \frac{1}{z}$$

$$= \frac{1}{1+a} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p^0_n(\theta) \frac{\theta^{\frac{k(k-1)}{2}}}{k!} \left(\tfrac{a}{1+a}\right)^k \left(\tfrac{1}{1+a}\right)^n z^{k+n} + \sum_{k=1}^{\infty} \frac{\theta^{\frac{k(k-1)}{2}}}{k!} \left(\tfrac{a}{1+a}\right)^k z^{k-1}$$

$$= \frac{1}{1+a} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} p^0_{n-k}(\theta) \frac{\theta^{\frac{k(k-1)}{2}}}{k!} \left(\tfrac{a}{1+a}\right)^k \left(\tfrac{1}{1+a}\right)^{n-k} z^n + \sum_{k=0}^{\infty} \frac{\theta^{\frac{k(k+1)}{2}}}{(k+1)!} \left(\tfrac{a}{1+a}\right)^{k+1} z^k$$

$$= \sum_{n=0}^{\infty} z^n \left(\tfrac{1}{1+a}\right)^{n+1} \sum_{k=0}^{n} p^0_{n-k}(\theta) \frac{\theta^{\frac{k(k-1)}{2}}}{k!} a^k + \sum_{n=0}^{\infty} \frac{\theta^{\frac{n(n+1)}{2}}}{(n+1)!} \left(\tfrac{a}{1+a}\right)^{n+1} z^n,$$

where we used formula (16) (applied to $z' = z/(1+a)$ and $a' = 0$) in the third step. This allows us to read off the $p^a_n(\theta)$ to get

$$p^a_n(\theta) = \left(\frac{1}{1+a}\right)^{n+1} \sum_{k=0}^{n} p^0_{n-k}(\theta) \frac{\theta^{\frac{k(k-1)}{2}}}{k!} a^k + \frac{\theta^{\frac{n(n+1)}{2}}}{(n+1)!} \left(\frac{a}{1+a}\right)^{n+1},$$

and using Corollary 5 shows the claim. $\qquad\square$

We can also extend this by duality to the case $\theta \geqslant 1$ and to the case $\theta < -1$ with $-1/\theta \leqslant a \leqslant -\theta$.

Corollary 7. *Let (ϕ_ℓ) be as defined in (14). Set $\bar{\theta} := 1/\theta$.*

(a) *Let $\theta \geqslant 1$ and $a \geqslant 0$. Then*

$$p_n^a(\theta) = \frac{1}{(1+a)^{n+1}} \sum_{k=0}^{n+1} \phi_{n+1-k}(\bar{\theta})(-a)^{n+1-k} \frac{\bar{\theta}^{k(k-1)/2}}{k!}. \tag{21}$$

(b) *Let $\theta < -1$ with $-1/\theta \leqslant a \leqslant -\theta$. Then*

$$p_n^a(\theta) = \left(\frac{-1}{1+a}\right)^{n+1} \sum_{k=0}^{n+1} \phi_{n+1-k}(\bar{\theta}) \frac{\bar{\theta}^{\frac{k(k-1)}{2}}}{k!}(-a)^k. \tag{22}$$

Proof. To prove the two parts, one can use the dualities (6) and (7), respectively, together with Corollary 6. □

5 The green region

In this section, we deal with the case $\theta \in [-1, 0]$ and $a \geqslant -1/\theta$. The idea is to reduce this case to the borderline $\theta \in [-1, 0]$ and $a = -1/\theta$ by conditioning the random variables to be $\geqslant 1/\theta$. We can then apply the formula from the blue region which also covers this borderline.

Lemma 7 (Equation (3) of Theorem 1). *Let $\theta \in [-1, 0]$ and $a \geqslant -1/\theta$. Then*

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \frac{1+\frac{a}{\theta}}{1+a} + \frac{E(\theta, \frac{-\bar{\theta}z}{1+a}) - E(\theta, \frac{-z}{1+a})}{z E(\theta, \frac{-z}{1+a})}. \tag{23}$$

Proof. We first observe that for $n \geqslant 1$,

$$p_n^a(\theta) = \mathbb{P}(\theta X_1 \leqslant X_2, \theta X_2 \leqslant X_3, \dots, \theta X_n \leqslant X_{n+1}, X_1 \geqslant \tfrac{1}{\theta}, \dots, X_{n+1} \geqslant \tfrac{1}{\theta}).$$

Indeed, since $\theta X_i \leqslant X_{i+1} \leqslant 1$, we get $X_i \geqslant \frac{1}{\theta}$ for all $1 \leqslant i \leqslant n$. Further, we have $X_{n+1} \geqslant \theta X_n \geqslant \theta \geqslant \frac{1}{\theta}$. As a consequence, conditioning on $X_i \geqslant \frac{1}{\theta}$, we rewrite

$$p_n^a(\theta) = \mathbb{P}(\theta \tilde{X}_1 \leqslant \tilde{X}_2, \theta \tilde{X}_2 \leqslant \tilde{X}_3, \dots, \theta \tilde{X}_n \leqslant \tilde{X}_{n+1}) \nu^{n+1},$$

where the $\tilde{X}_i$ are uniform on $[-\frac{1}{\theta}, 1]$ and $\nu = \mathbb{P}(X_i \geqslant \frac{1}{\theta}) = \frac{1-\frac{1}{\theta}}{1+a}$. As it turns out, the case $a = -\frac{1}{\theta}$ is a boundary case of the blue region applicable to the $\tilde{X}_i$, so we can continue the computations as follows:

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = 1 - \nu + \nu \sum_{n=0}^{\infty} p_n^{-\frac{1}{\theta}}(\theta)(\nu z)^n,$$

where, using Lemma 6,

$$\sum_{n=0}^{\infty} p_n^{-\frac{1}{\theta}}(\theta)(\nu z)^n = \frac{E\left(\theta, \frac{-\nu z}{\theta(1-\frac{1}{\theta})}\right) - E\left(\theta, \frac{-\nu z}{1-\frac{1}{\theta}}\right)}{\nu z E\left(\theta, -\frac{\nu z}{1-\frac{1}{\theta}}\right)}.$$

Combining the last two displays exactly corresponds to (23). □

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A simple corollary is a formula for the persistence exponent.

Corollary 8. *Let $\theta \in [-1,0]$ and $a \geqslant -1/\theta$. The persistence exponent λ is given by the inverse $\lambda = 1/z$ of the smallest positive root z of the equation $E(\theta, \frac{-z}{1+a}) = 0$.*

We can also prove a combinatorial representation of the persistence probabilities in terms of the quantities ϕ_ℓ from (14).

Corollary 9. *For $\theta \in [-1,0]$ and $a \geqslant -1/\theta$ we have $p_0^a(\theta) = 1$ and, for $n \geqslant 1$,*

$$p_n^a(\theta) = \sum_{k=0}^{n+1} \frac{\theta^{k(k-3)/2}}{k!} \phi_{n+1-k}(\theta) \frac{(-1)^{n+1}}{(1+a)^{n+1}},$$

where the (ϕ_ℓ) are defined in (14).

Proof. By Lemma 7 and Lemma 5,

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = 1 + \frac{\bar{\theta}-1}{1+a} + z^{-1} \left(\frac{E(\theta, \frac{-\bar{\theta}z}{1+a})}{E(\theta, \frac{-z}{1+a})} - 1 \right)$$

$$= 1 + \frac{\bar{\theta}-1}{1+a} + z^{-1} \left(\sum_{k=0}^{\infty} \frac{\theta^{k(k-1)/2}}{k!} \left(\frac{-\bar{\theta}z}{1+a} \right)^k \cdot \sum_{n=0}^{\infty} \phi_n(\theta) \left(\frac{-z}{1+a} \right)^n - 1 \right)$$

$$= 1 + \frac{\bar{\theta}-1}{1+a} + z^{-1} \left(\sum_{k=0}^{\infty} \frac{\theta^{k(k-1)/2}}{k!} \left(\frac{-\bar{\theta}z}{1+a} \right)^k \cdot \sum_{n=k}^{\infty} \phi_{n-k}(\theta) \left(\frac{-1}{1+a} \right)^{n-k} z^{n-k} - 1 \right)$$

$$= 1 + \frac{\bar{\theta}-1}{1+a} + z^{-1} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\theta^{k(k-1)/2}}{k!} \bar{\theta}^k \phi_{n-k}(\theta) \frac{(-1)^n}{(1+a)^n} \cdot z^n - 1 \right)$$

$$= 1 + \frac{\bar{\theta}-1}{1+a} + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{\theta^{k(k-3)/2}}{k!} \phi_{n-k}(\theta) \frac{(-1)^n}{(1+a)^n} \cdot z^{n-1}.$$

From here, we can read off the $p_n^a(\theta)$. $\qquad\square$

6   The yellow region

In this section, we deal with the case $a \in [0,1]$ and $\theta \in [-1,-a]$. The idea is to relate the generating function of the persistence probabilities in this region to the one in the green region via the duality (12).

Lemma 8 (Equation (4) of Theorem 1)**.** *If $a \in (0,1)$ and $\theta \in [-1,-a]$, then with $\mu := \frac{1+\frac{a}{\theta}}{1+a}$*

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \frac{1}{z} \left(\frac{1}{\frac{E(\theta, \frac{az}{\theta(1+a)})}{E(\theta, \frac{az}{1+a})} - \mu z} - 1 \right). \qquad (24)$$

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Proof. This will follow directly from the duality stated in Corollary 1, combined with the expression provided in Lemma 7 for the green region. Indeed, denoting by $P_G^a(\theta)$ the formula (23) in the green region, we have, using Corollary 1,

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \frac{P_G^{\frac{1}{a}}(-z)}{1 - z P_G^{\frac{1}{a}}(-z)} = \frac{1}{z} \left(\frac{1/z}{1/z - P_G^{\frac{1}{a}}(-z)} - 1 \right).$$

On the other hand, a computation starting from (23) shows that

$$P_G^{\frac{1}{a}}(-z) = \mu - \frac{1}{z} \left(\frac{E(\theta, \frac{az}{\theta(1+a)})}{E(\theta, \frac{az}{1+a})} - 1 \right),$$

and we easily conclude (24). □

A simple corollary is a formula for the persistence exponent.

Corollary 10. *Let $a \in [0,1]$ and $\theta \in [-1, -a]$. Recalling our notation $\mu = \frac{1 + \frac{a}{\theta}}{1+a}$, the persistence exponent λ is given by the inverse $\lambda = 1/z$ of the smallest positive root z of the equation $\frac{E(\theta, \frac{az}{\theta(1+a)})}{E(\theta, \frac{az}{1+a})} = \mu z$.*

Contrary to the blue and yellow region, we did not find an explict expression of the $(p_n^a(\theta))$ in terms of the combinatorial quantities $(\phi_\ell(\theta))$ from Lemma 5.

7   The orange region

First, we deal with the case $-b := a \leqslant 0$ and $1 \leqslant \theta \leqslant 1/b$. Note that, contrary to the other regions, we start with the computation for $\theta > 1$ and then reduce it to the result for $\theta \in [-1, 1]$. This is due to the fact that the probabilities are much simpler for $\theta > 1$.

In fact, for $b < 0$, the persistence probabilities become zero at some point. Even more, the remaining positive persistence probabilities can be written down in a very simple form. The case $a = b = 0$ is a special case of this computation.

Lemma 9. *Let $a \in (-1, 0]$ and $1 \leqslant \theta \leqslant -1/a$. We set $b := -a$. Then*

$$p_n^a(\theta) = \frac{1}{(1-b)^{n+1}} \frac{\theta^{(n+1)n/2}}{(n+1)!} \left(\frac{1}{\theta^n} - b \right)_+^{n+1}, \tag{25}$$

where $x_+ := \min(x, 0)$. In particular,

- *If $b > 0$ then we have*

$$p_n^a(\theta) = 0, \qquad \text{for all } n \text{ such that } \theta^n b \geqslant 1.$$

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- *If $b = 0$ then the formula simplifies to*

$$p_n^0(\theta) = \frac{\theta^{-(n+1)n/2}}{(n+1)!}.$$

We remark that, for $b > 0$, due to the fact that $\theta b < 1$ and $\theta > 1$, we know that $\theta^n b \geqslant 1$ for all $n \geqslant n_0$ for some finite $n_0 > 1$.

Further, given the simple form of the persistence probabilities, one can easily compute the generating function, e.g. for $b = 0$:

$$\sum_{n=0}^{\infty} p_n^0(\theta) z^n = \frac{1}{z}(E(\theta^{-1}, z) - 1). \tag{26}$$

Another remark is that the formula (26) can also be obtained from Lemma 6 together with the duality (6): Fixing $\theta > 1$ and letting $a \to 0$, we have

$$\sum_{n=0}^{\infty} p_n^a(\theta) z^n = \sum_{n=0}^{\infty} p_n^{1/a}(1/\theta) z^n = \frac{E(\theta^{-1}, \frac{z}{a+1}) - E(\theta^{-1}, -\frac{az}{1+a})}{z E(\theta^{-1}, -\frac{az}{1+a})} \to \frac{1}{z}(E(\theta^{-1}, z) - 1).$$

Concerning the persistence exponent, we notice that the persistence probabilities in Lemma 9 decay superexponentially.

Proof of Lemma 9. Note that if $X_1 > 1/\theta$ then $\theta X_1 \leqslant X_2$ implies that $X_2 > 1$, which is not possible. Similarly, $X_1 > 1/\theta^{k-1}$ and the relations $\theta^{k-1} X_1 \leqslant \theta^{k-2} X_2 \leqslant \ldots \leqslant \theta X_{k-1} \leqslant X_k$ imply $X_k > 1$. On the other hand, if $x_k \geqslant b$ then $\theta x_k \geqslant \theta b \geqslant b$.

First, the above reasoning implies that $p_n^a(\theta) = 0$ for $b \geqslant 1/\theta^n$. Thus, for the rest of the proof, we can concentrate on the case $b \leqslant 1/\theta^n$.

Taking the above considerations into account, we have

$$p_n^a(\theta) = \mathbb{P}(\theta X_1 \leqslant X_2, \ldots, \theta X_n \leqslant X_{n+1})$$
$$= \frac{1}{(1-b)^{n+1}} \int_b^{1/\theta^n} \int_{\theta x_1}^{1/\theta^{n-1}} \cdots \int_{\theta x_{n-1}}^{1/\theta} \int_{\theta x_n}^{1} 1 \mathrm{d}x_{n+1} \mathrm{d}x_n \ldots \mathrm{d}x_2 \mathrm{d}x_1.$$

For simplicity of notation, set

$$q_n^a(\theta) := (1-b)^{n+1} p_n^a(\theta)$$
$$= \int_b^{1/\theta^n} \int_{\theta x_1}^{1/\theta^{n-1}} \cdots \int_{\theta x_{n-1}}^{1/\theta} \int_{\theta x_n}^{1} 1 \mathrm{d}x_{n+1} \mathrm{d}x_n \ldots \mathrm{d}x_2 \mathrm{d}x_1 \tag{27}$$
$$= \int_b^{1/\theta^n} q_{n-1}^{\theta x_1}(\theta) \, \mathrm{d}x_1,$$

and it only remains to check the claim on the lemma by induction using the latter recursion. For $n = 1$, we have due to (27)

$$q_n^a(\theta) = \int_b^{1/\theta} \int_{\theta x_1}^{1} 1 \mathrm{d}x_2 \mathrm{d}x_1 = \int_b^{1/\theta} (1 - \theta x_1) \mathrm{d}x_1 = \frac{1}{\theta} - b - \frac{\theta}{2}\left(\frac{1}{\theta^2} - b^2\right) = \frac{\theta}{2}\left(\frac{1}{\theta} - b\right)^2,$$

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in accordance with (25). If (25) is true for $n-1$, we get from the recursion (using that $\frac{1}{\theta^{n-1}} \geqslant \theta x_1$) that

$$
\begin{aligned}
q_n^a(\theta) &= \int_b^{1/\theta^n} q_{n-1}^{\theta x_1}(\theta) \mathrm{d}x_1 \\
&= \int_b^{1/\theta^n} \frac{\theta^{(n-1)n/2}}{n!} \left(\frac{1}{\theta^{n-1}} - \theta x_1\right)^n \mathrm{d}x_1 \\
&= \frac{\theta^{(n-1)n/2}}{n!} \int_b^{1/\theta^n} \theta^n \left(\frac{1}{\theta^n} - x_1\right)^n \mathrm{d}x_1 \\
&= \frac{\theta^{(n+1)n/2}}{(n+1)!} (-1) \left(\frac{1}{\theta^n} - x_1\right)^{n+1} \Bigg|_{x_1=b}^{1/\theta^n} \\
&= \frac{\theta^{(n+1)n/2}}{(n+1)!} \left(\frac{1}{\theta^n} - b\right)^{n+1},
\end{aligned}
$$

as required in the case $b \leqslant 1/\theta^n$. $\qquad\square$

We now treat the case $\theta \in (0, 1]$.

Lemma 10 (Equation (5) of Theorem 1). *Let $\theta \in (0, 1]$ and $0 \leqslant -a =: b \leqslant \theta$. Assume that $p \geqslant 1$ is such that $\theta^{p+1} < b \leqslant \theta^p$. Then the generating function of the persistence probabilities is given by*

$$
\sum_{n=0}^\infty p_n^a(\theta) z^n = \frac{F(z)}{1 - zF(z)}, \tag{28}
$$

where

$$
F(z) := \sum_{i=0}^p (-1)^i \frac{(\theta^i - b)^{i+1}}{(i+1)!\theta^{i(i+1)/2}(1-b)^{i+1}} z^i = \sum_{i=0}^\infty (-1)^i \frac{(\theta^i - b)_+^{i+1}}{(i+1)!\theta^{i(i+1)/2}(1-b)^{i+1}} z^i. \tag{29}
$$

In particular, the generating function $\sum_{n=0}^\infty p_n^a(\theta) z^n$ is a piecewise rational function.

Proof. The lemma follows directly from the duality (9) and Lemma 9. $\qquad\square$

We remark that the function F is nothing else but the generating function of the persistence probabilities $p_n^a(1/\theta)$ computed from (25).

A simple corollary is a formula for the persistence exponent.

Corollary 11. *Let $\theta \in (0, 1]$ and $0 \leqslant -a =: b \leqslant \theta$. The persistence exponent λ is given by the inverse $\lambda = 1/z$ of the smallest positive root z of the equation*

$$
zF(z) = 1, \tag{30}
$$

where F is the function from (29).

We remark that for $b > 0$ the equation (30) is a polynomial equation for the persistence exponent λ (the degree of the polynomial is $p+1$ if $\theta^p < b \leqslant \theta^p$).

Acknowledgments

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