Equivalent differential equations of order one

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Abstract

The notions of equivalence and strict equivalence for order one differential equations of the form $f(y', y, z) = 0$ are introduced. The more explicit notion of strict equivalence is applied to examples and questions concerning autonomous equations and equations having the Painlevé property. The order one equation f determines an algebraic curve X over $\mathbb{C}(z)$. If X has genus 0 or 1, then it is difficult to verify strict equivalence. However, for higher genus strict equivalence can be tested by an algorithm sketched in the text. For autonomous equations, testing strict equivalence and the existence of algebraic solutions are shown to be algorithmic.

Keywords. Ordinary differential equations: algebraic curves, local behavior of solutions, normal forms, Painlevé property, algebraic solutions.

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1 Equivalence

For an irreducible polynomial $f := f(S,T,z) \in \mathbb{C}(z)[S,T]$, we consider the order one differential equation $f(y',y,z) = 0$, where $y' = \frac{dy}{dz}$. The special case that S is not present in f is not really a differential equation and the solutions are algebraic over $\mathbb{C}(z)$. The other special case, namely T is not present in f, is still a differential equation. The solutions are the integrals of finitely many functions which are algebraic over $\mathbb{C}(z)$ (compare [Bro], [Tra]). We will exclude these special cases.

In the sequel we will also consider finite extensions K of $\mathbb{C}(z)$, equipped with the unique extension of $\frac{d}{dz}$ to K (which we also denote as $\frac{d}{dz}$). Moreover, we will suppose that $f \in K[S,T]$ is absolutely irreducible.

For some order one differential equations, like the Riccati equation $y' = ay^2 + by + c$, it is easy to describe the solutions. For general f it is difficult to find any information on the solutions and equivalence of equations is a basic theme.

An intuitive way of describing that two such order one differential equation f_1 and f_2 are equivalent is the existence of an algebraic procedure to obtain from a solution of f_1 a solution (or finitely many solutions) of f_2 and vice versa.

In order to make this more precise we have to define what a solution y of $f \in K[S,T]$ is. First we observe that a solution y of $f(y',y,z) = 0$ which is also a solution of $\frac{\partial f}{\partial S}(y',y,z) = 0$, is algebraic over K since the ideal $(f, \frac{\partial f}{\partial S}) \subset K[S,T]$ has finite codimension as a K-vector space. These solutions are easily computed. Therefore we restrict to solutions y of f such that $\frac{\partial f}{\partial S}(y',y,z) \neq 0$.

We consider the algebra $K[s,t] := K[S,T]/(f)$ and try to make this into a differential algebra by the derivation $z' = 1, t' = s$. Then the derivative s' of s is obtained by differentiation of $0 = f(s,t,z)$. Thus

$$0 = s' \cdot \frac{\partial f}{\partial s} + s \cdot \frac{\partial f}{\partial t} + \frac{\partial f}{\partial z},$$

and we will restrict to the case that

$$d := \frac{\partial f}{\partial s}$$

is invertible. Then $K[s,t,\frac{1}{d}]$ is a differential algebra. We note that $\frac{\partial f}{\partial S}$ is called the 'separant' and that the above differential algebra coincides with the 'generic solution' of f in the terminology of [Ri], p. 129-131. In [K], §16 of Chapter IV, related material is considered.

A solution of f is supposed to be algebraic over the field $Mer(U)$ of the mero-morphic functions on the universal covering U of an open connected subset of the Riemann surface of K. Let $Mer(U)^a$ denote the algebraic closure of the field $Mer(U)$. The differentiation $\frac{d}{dz}$ on K extends uniquely to a differentiation on $Mer(U)^a$. A *solution of f* is a K-linear homomorphism

$$\phi : K[s,t,\frac{1}{d}] \to Mer(U)^a$$

commuting with differentiation (and $y := \phi(t)$ is the actual solution).

A variant of the above is the notion of *local solution*. This is a K-linear differential homomorphism

$$\phi : K[s,t,\frac{1}{d}] \to \mathbb{C}(\{v^{1/m}\}).$$

The latter is the field of the convergent Laurent series in the variable $v^{1/m}$, where v is a local variable of a point of the Riemann surface of K and $m \in \mathbb{Z}_{\geq 1}$. Of course a local solution ϕ extends to a solution in $Mer(U)^a$, where U is a small disk around a point of the Riemann surface of K. On the other hand a solution in some $Mer(U)^a$ induces local solutions at the points of U. In the following, the precise definition of solution does not play a role. However, in contrast to [Ri] and [K], we have chosen for a concrete definition of solution.

In the case that $K[s,t,\frac{1}{d}]$ has only trivial differential ideals, its field of fractions has $\mathbb{C}$ as field of constants (see [vdP-S]) and $\ker\phi = 0$. If $\ker\phi \neq 0$, then $\ker\phi$ is a maximal ideal of $K[s,t,\frac{1}{d}]$. The solution $y = \phi(t)$ is algebraic over $\mathbb{C}(z)$ and is considered as an element of $Mer(U)^a$. There are very few equations admitting an algebraic solution.

It seems to be an open problem whether there exists an algorithm testing the existence of (and computing) algebraic solutions for a first order differential equation.
Let f be an order 1 equation. For positive integers d,n one considers algebraic elements y satisfying an equation $a_d y^d + a_{d-1}y^{d-1} + \cdots + a_0 = 0$, where the $a_d, \ldots, a_0$ are polynomials of degree $\leq n$. The coefficients of the a_j are seen as variables. Differentiation of this identity yields an expression for y'. The substitution $f(y',y,z) = 0$ produces a set of polynomial equations in many variables (over a computable subfield of $\mathbb{C}$). Gröbner theory provides an

algorithm for solving this. Thus the problem of finding algebraic solutions is 'recursive enumerable'. Missing for a true algorithm are a priori estimates for d, n.

For special cases, there are estimates for d, n. Here are some examples. The solutions of the autonomous equation $y' = R(y)$, with $R(y) \in \mathbb{C}(y)$, satisfy $\int \frac{dy}{R(y)} = z + c$. The Risch algorithm finds the algebraic solutions (if any). An equation like $(y')^2 = y^5 + 1$ yields Abelian integrals which are transcendental.

For a Riccati equation $y' + ay^2 + by + c = 0$ with $a, b, c \in \mathbb{C}(z)$, Kovacic's algorithm tests the existence and computes possible algebraic solutions. This is done by computing local solutions at the singular points and the observation that a solution y can only have poles of order one and integer residue at the non singular points of the equation. The above equation has PP, the Painlevé property (see § 2 below). For every equation with PP there is an algorithm for finding algebraic solutions.

In contrast to the above, we do not know whether a simple equation like $y' = y^3 + z$ over $\mathbb{C}(z)$ has algebraic solutions. The local solutions are:
For $z = a \neq \infty$ there is a holomorphic solution $y \in \mathbb{C}\{z - a\}$, depending on the initial value $y(a)$. Moreover there is a ramified meromorphic solution $y = \sum_{n \geq -1} a_n (z - a)^{n/2}$ in $\mathbb{C}(\{(z - a)\})$, depending on a_{-1} and $a_{-1}^2 = \frac{-1}{2}$.
For $z = \infty$ and with $t := \frac{1}{z}$ the equation reads $\frac{dy}{dt} = -t^2 y^3 - t^3$. The solutions are $y = \sum_{n \geq -1} c_n t^{n/3}$ in $\mathbb{C}(\{t^{1/3}\})$, depending on c_{-1} and $c_{-1}^3 = -1$.

An algebraic solution y has to be ramified at $z = \infty$ of order 3 (and thus y is not rational) and is ramified at some more points with ramification of order 2. However we have no idea what the other ramification points for y could be and what the degree of y over $\mathbb{C}(z)$ could be.

A criterion for the existence of *generic algebraic solutions* is proposed in [A-C-F-G]. For autonomous equations the above criterion leads to an algorithm. For a first order differential equation f, a generic algebraic solution is a 1-parameter family $\{y_c\}$ of algebraic solutions such that f is the minimal equation for this family. For example, the equation $y' = y^5$ has the generic solution $y_c^4 = \frac{-1}{4z+c}$. First order equations with a generic algebraic solution are very rare.

Definition 1.1 (*Equivalent equations*). An *equivalence* between equations f_1 and f_2 is given by a $\mathbb{C}(z)$-linear differential isomorphism $\psi : F_1 \rightarrow F_2$, where for

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$j = 1, 2$, the differential field F_j is a finite extension of the field of fractions of $\mathbb{C}(z)[S, T, \frac{1}{d_j}]/(f_j)$.

It is easily seen that the above definition induces an equivalence relation. Let f_1 and f_2 be equivalent. Fix ψ. Let y be a solution for f_1, given by

$$\phi : \mathbb{C}(z)[S, T, \frac{1}{d_1}]/(f_1) \to Mer(U)^a.$$

Then ϕ extends to the field of fractions of $\mathbb{C}(z)[S, T, \frac{1}{d_1}]/(f_1)$ and has finitely many extensions $\phi_1, \ldots, \phi_r$ to F_1. The restriction of

$$F_2 \xrightarrow{\psi} F_1 \xrightarrow{\phi_j} Mer(U)^a$$

to $\mathbb{C}(z)[S, T, \frac{1}{d_2}]/(f_2)$ is a solution of f_2.

We conclude that the above definition of equivalence is a way to make the intuitive notion explicit. It seems rather difficult to decide for explicit f_1 and f_2 whether they are equivalent. Therefore we introduce the following notion.

Definition 1.2 (*Strictly equivalent equations*)**.** The equations f_1 and f_2 are *strictly equivalent* if there is a finite extension K of $\mathbb{C}(z)$ and a K-linear differential isomorphism ψ between the fields of fractions of $K[S, T, \frac{1}{d_1}]/(f_1)$ and $K[S, T, \frac{1}{d_2}]/(f_2)$.

Remarks 1.3. (i). The ψ in Definition 1.2 need not be unique. Indeed, two distinct ψ's differ by a K-linear differential automorphism of the field of fractions of $K[S, T, \frac{1}{d_1}]/(f_1)$. The group of the differential automorphism induce a group of permutations of the solutions of f_1.
(ii). We note that Definitions 1.1 and 1.2 extend in an obvious way to equations $f \in K[S, T]$, where K is any differential field with field of constants $\mathbb{C}$.
(iii). In the sequel we will study strict equivalence for order one differential equations using well known properties of algebraic curves.

2 The Painlevé property

An ordinary differential equation on the complex plane is said to have the *Painlevé property* (PP for short) if there are no other moving singularities than poles.

The Painlevé property for order one equations has been analysed in detail in [Mun-vdP]. The reasoning and the results are as follows.

(1). Observation: *If the order one equation f has PP, then f has only finitely many branched solutions.*
A branched solution is a solution $\phi : \mathbb{C}(z)[s,t,\frac{1}{d}] \to \mathbb{C}(\{(z-a)^{1/m}\})$ with $m > 1$ and such that $y = \phi(t)$ is not contained in $\mathbb{C}(\{z-a\})$.
(2). The field of fractions F of $\mathbb{C}(z)[s,t,\frac{1}{d}]$ is the function field of a smooth projective curve X over $\mathbb{C}(z)$. Let D denote the differentiation on F.
If the equation f has only finitely many branch points, then every local ring $O_{X,Q}$ (where Q is a closed point) is invariant under D.
(3). *Suppose that every local ring $O_{X,Q}$ is invariant under D. Then there exists a finite extension $K \supset \mathbb{C}(z)$ and a smooth, connected curve X_0 over $\mathbb{C}$, such that $K \times_{\mathbb{C}(z)} X \cong K \times_{\mathbb{C}} X_0$. Moreover:*
(i) If $X_0 \cong \mathbb{P}^1_{\mathbb{C}}$, then f is strictly equivalent to $y' = a_0 + a_1 y + a_2 y^2$ with $a_0, a_1, a_2 \in K$, not all zero.
(ii) If X_0 has genus 1 and equation $y^2 = x^3 + ax + b$, then f is strictly equivalent to $(y')^2 = h \cdot (y^3 + ay + b)$ for some $h \in K^*$.
(iii) If X_0 has genus ≥ 2, then f is strictly equivalent to $y' = 0$.
(4). Finally, the cases (i)–(iii) have PP.

From (1)–(4) one deduces the following.

Proposition 2.1. *Let f_1 and f_2 be strictly equivalent. Then f_1 has PP if and only if f_2 has PP.*

An order one equation f is called *autonomous* if f is an irreducible element of $\mathbb{C}[S,T]$. A rather difficult question is *whether a given f is strictly equivalent to an autonomous equation.* Let X denote the smooth connected curve over $\mathbb{C}(z)$ such that its function field is the field of fractions of $\mathbb{C}(z)[s,t,\frac{1}{d}]$. We will call f *semi-autonomous* if $K \times_{\mathbb{C}(z)} X \cong K \times_{\mathbb{C}} X_0$ for some curve X_0 over $\mathbb{C}$ and some finite extension K of $\mathbb{C}(z)$. The curve X over $\mathbb{C}(z)$ can be interpreted as a surface with a projection to $\mathbb{P}^1_{\mathbb{C}}$. In other words, X has the interpretation of a family of curves over $\mathbb{C}$. The condition 'semi-autonomous' is identical to 'X is an isotrivial family of curves'.
In the next sections we intend to treat the following items:
(i) The existence of an algorithm deciding whether two curves X_1, X_2 over a finite extension K of $\mathbb{C}(z)$ become isomorphic after a finite extension L of K. This includes deciding whether a given first order equation is semi-autonomous.
(ii) The existence of an algorithm deciding strict equivalence between two first order differential equations in case the genus is ≥ 2.

(iii) The question whether strict equivalence is undecidable for the cases of genus 0 and 1.

3 Autonomous equations

We associate to an irreducible autonomous equation $f(y',y) = 0$ (we assume that both y and y' are present in f) the pair (X,D) where the complete, irreducible, smooth curve X has function field $\mathbb{C}(y_1,y_0)$ with equation $f(y_1,y_0) = 0$ and D is the meromorphic vector field on X determined by $D(y_0) = y_1$.

Lemma 3.1. *Every pair (X,D), consisting of a curve $X/\mathbb{C}$ (complete, irreducible, smooth) and a non zero meromorphic vector field D on X, is associated to some autonomous equation $f(y',y) = 0$.*

Proof. Let $g \in \mathbb{C}(X)$ satisfy $D(g) \neq 0$. Choose a closed point $x \in X$ such that g has no pole at x, $ord_x(g - g(x)) = 1$ and $ord_x(D(g)) = 0$. Let p denote a local parameter at x. Then $\widehat{O}_{X,x} = \mathbb{C}[[p]]$ and $ord_x(D(p)) = 0$. Let ℓ be a prime number such that $\ell > 2 \cdot \text{genus}(X) + 2$ and let $f \in \mathbb{C}(X)$ have a pole of order ℓ at x and no further poles. Then $[\mathbb{C}(X) : \mathbb{C}(f)] = \ell$. If $D(f) \notin \mathbb{C}(f)$, then $\mathbb{C}(f,D(f)) = \mathbb{C}(X)$ since ℓ is a prime number.

Suppose that $D(f) \in \mathbb{C}(f)$. Then $\frac{D(f)}{f} \in \mathbb{C}(f) \subset \mathbb{C}((\frac{1}{f})) = \mathbb{C}((p^\ell)) \subset \mathbb{C}((p))$. This contradicts the fact that $ord_x(\frac{D(f)}{f}) = -1$. $\qquad\square$

Remark 3.2. An irreducible order one equation $f(y',y,z) = 0$ over a finite field extension K of $\mathbb{C}(z)$ induces a pair (X,D) of a curve X over K and a derivation D of the function field of X/K. The proof of Lemma 3.1 extends to this non autonomous case. The statement is: For a given pair (X,D) over K, there exists a finite extension $\tilde{K}$ of K and an irreducible order one equation $f(y',y,z) = 0$ over $\tilde{K}$ which induces $\tilde{K} \times_K X$ equipped with the unique extension of D. $\qquad\square$

By (X,D) we denote a pair as in Lemma 3.1. Further for any finite extension K of $\mathbb{C}(z)$ we denote by $K \times (X,D)$ the curve $K \times_\mathbb{C} X$ with function field $K \otimes_\mathbb{C} \mathbb{C}(X)$ equipped with the derivation D^+ defined by $D^+ = \frac{d}{dz}$ on K and $D^+ = D$ on $\mathbb{C}(X)$. We note that D^+ is not a meromorphic vector field since it is not zero on K.

Lemma 3.3. *Let $\phi : K \times (X_1,D_1) \to K \times (X_2,D_2)$ be an isomorphism. Then there exists an isomorphism $(X_1,D_1) \to (X_2,D_2)$.*

Proof. The isomorphism $\phi : K \times X_1 \to K \times X_2$ induces an isomorphism $\phi_1 : spec(R) \times X_1 \to spec(R) \times X_2$ for some finitely generated $\mathbb{C}$-algebra R with field of fractions K. After dividing by a maximal ideal of R we find an isomorphism $X_1 \to X_2$. In the sequel we identify X_1 and X_2 with some X. It is given that some automorphism ϕ of $K \times X$ has the property $D_2^+ = \phi D_1^+ \phi^{-1}$. We have to show that there exists an automorphism ψ of X with $D_2 = \psi D_1 \psi^{-1}$.

If the genus of X is ≥ 2, then $\overline{K} \times X$ and X have the same finite group of automorphisms and there is nothing to prove.

Suppose that X has genus zero. Write $\mathbb{C}(X) = \mathbb{C}(y)$. On the field $K(y)$ we consider two derivations: $\frac{d}{dy}$ with $\frac{d}{dy}(y) = 1$ and $\frac{d}{dy}$ is zero on K; further $\frac{d}{dz}$ defined by $\frac{d}{dz}(z) = 1$ and $\frac{d}{dz}(y) = 0$. Let $D_j(y) = f_j(y) \in \mathbb{C}(y)$, then $D_j^+ = \frac{d}{dz} + f_j(y)\frac{d}{dy}$ for $j = 1, 2$. Suppose $D_2^+ = \phi^{-1}D_1^+\phi$ where $\phi(y) = \frac{Ay+B}{Cy+D}$ with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(K)$. One computes the identity

$$f_2(y) = (A'C - AC')(Dy - B)^2 + (A'D - AD' + B'C - BC')(Dy - B)(-Cy + A) +$$

$$(B'D - BD')(-Cy + A)^2 + (-Cy + A)^2 + (-Cy + A)^2 f_1(\frac{Dy - B}{-Cy + A}).$$

A pole p of $f_1\frac{d}{dy}$ yields a pole $\phi(p)$ of $f_2\frac{d}{dy}$ and so $\phi(p) \in \mathbb{C} \cup \{\infty\}$. If f_1 has at least three poles, then ϕ is an automorphism of $\mathbb{P}^1_{\mathbb{C}}$ and we can take $\psi = \phi$.

If $f_1\frac{d}{dy}$ has two poles, then the same holds for $f_2\frac{d}{dy}$. We may suppose that these poles are 0 and ∞ and that ϕ has 0 and ∞ as fixed points. Then $D = A^{-1}$, $B = C = 0$ and an explicit calculation shows that again $\phi \in \mathrm{PSL}_2(\mathbb{C})$. A similar calculation can be made for the case that $f_1\frac{d}{dy}$ has at most one pole.

Suppose that the genus of X is one and consider X as an elliptic curve with function field $\mathbb{C}(X) = \mathbb{C}(x, y)$ with relation $y^2 = x^3 + ax + b$. Then $y\frac{d}{dx}$ is the standard invariant derivation on $\mathbb{C}(x, y)$. Let $D_j = f_j y\frac{d}{dx}$ with $f_j \in \mathbb{C}(x, y)$. We extend $y\frac{d}{dx}$ to $K(x, y)$ by $y\frac{d}{dx}$ is zero on K and introduce $\frac{d}{dz}$ on $K(x, y)$ by $\frac{d}{dz}(z) = 1$ and $\frac{d}{dz}$ is zero on $\mathbb{C}(x, y)$. Then $D_j^+ = \frac{d}{dz} + f_j y\frac{d}{dx}$. We are given $D_2^+ = \phi D_1^+ \phi^{-1}$ and want to prove that there is an automorphism ψ of $\mathbb{C}(x, y)$ with $D_2 = \psi D_1 \psi^{-1}$. We may suppose that ϕ is a translation over the K-valued point (x_0, y_0) of X.

Now $\frac{d}{dz} + f_2 y\frac{d}{dx} = \phi\frac{d}{dz}\phi^{-1} + \phi f_1 y\frac{d}{dx}\phi^{-1}$. Now $\phi f_1 y\frac{d}{dx}\phi^{-1} = \phi(f_1) \cdot y\frac{d}{dx}$ and $\phi\frac{d}{dz}\phi^{-1} = \frac{d}{dz} - \frac{x_0'}{y_0}y\frac{d}{dx}$ (the last formula we found using an explicit Maple calculation). Suppose f_1 has a pole p. Then f_2 has a pole $\phi(p)$, since $\frac{x_0'}{y_0}y\frac{d}{dx}$ has no poles.

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The points p and $\phi(p)$ belong to $X(\mathbb{C})$ and therefore the translation ϕ is defined over $\mathbb{C}$. On the other hand, if f_1 has no poles, then the same holds for f_2 and $c := f_2 - f_1$ is a constant. Suppose $c \neq 0$. Then $(x_0')^2 = c^2 y_0^2 = c^2(x_0^3 + ax_0 + b)$. The non constant solutions of this Weierstrass equation are transcendental (since they are doubly periodic), contradicting the algebraicity of x_0. □

By 3.1 and 3.3, the set of the strict equivalence classes of autonomous first order equations coincides with the set of the equivalence classes of pairs (X, D). We sketch a proof of the statement: *There exists an algorithm deciding whether two pairs (X_j, D_j), $j = 1, 2$ are equivalent.*

In the next sections we will show that there is an algorithm deciding whether two curves of the same genus over a fixed algebraically closed field of characteristic zero are isomorphic.

This reduces the problem to the case $X := X_1 = X_2$ and deciding whether there exists an automorphism ψ of X such that $D_2 = \psi D_1 \psi^{-1}$. If the genus of X is ≥ 2, then the finite group of automorphisms of X is computable and this finishes this case.

Suppose that X has genus ≤ 1, then the automorphism group of X is infinite. However, the equality $D_2 = \psi D_1 \psi^{-1}$ implies that ψ sends the divisor of D_1 to the divisor of D_2. One easily verifies that there is an algorithm for deciding the latter condition. Moreover a meromorphic vector field D is determined, up to a constant, by its divisor.

Lemma 3.4. *Let the autonomous equation f induce the pair (X, D). Then $f(y', y) = 0$ has an algebraic solution if and only if $\mathbb{C}(X)$ contains an element t with $D(t) = 1$.*

Proof. Let $t \in \mathbb{C}(X)$ satisfy $D(t) = 1$. The differential isomorphism $\phi : \mathbb{C}(t) \to \mathbb{C}(z)$, which sends t to $z + c$ (any constant c) extends to a differential embedding of $\mathbb{C}(X)$ into the algebraic closure of $\mathbb{C}(z)$. In particular, this produces an algebraic solution for the equation $f(y', y) = 0$.

On the other hand, suppose that an algebraic solution y exists. This induces a differential embedding of $\mathbb{C}(X)$ into the algebraic closure of $\mathbb{C}(z)$. Thus z is an algebraic solution of the inhomogeneous differential equation $t' = 1$ over the differential field $\mathbb{C}(X)$. Since the differential Galois group of the equation $t' = 1$ is a finite algebraic subgroup of the additive group $\mathbb{G}_a$ one has $z \in \mathbb{C}(X)$. □

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We note that Lemma 3.4 is essentially present in [A-C-F-G].

An algorithm for algebraic solutions of the autonomous equation $f(y',y) = 0$.
Let the pair (X,D) be induced by f. According to Lemma 3.4, it suffices to produce an algorithm for finding a solution of $D(t) = 1$ with $t \in \mathbb{C}(X)$. Consider a closed point $x \in X$ with local parameter p. Then $\widehat{O}_{X,x} = \mathbb{C}[[p]]$.

A local solution at x has the form $t = a_k p^k + a_{k+1} p^{k+1} + \cdots \in \mathbb{C}((p))$ and $D(t) = (ka_k p^{k-1} + \cdots)D(p) = 1$.

If $D(p)$ has no pole or zero, then $t = a_0 + a_1 p + \cdots$ with $a_1 \neq 0$.

If $D(p)$ has a zero, $D(p) = b_k p^k + \cdots$, $b_k \neq 0$, $k \geq 1$, then $k = 1$ is not possible and for $k > 1$ one has t has a pole of order $k - 1$.

If $D(p)$ has a pole of order $-k$, then t has a zero of order $k + 1$.

It follows that a possible t with $D(t) = 1$ lies in $H^0(X,L)$ for a known line bundle L. Testing $D(t) = 1$ for the elements of $H^0(X,L)$ is done by using the Coates algorithm [Co].

4   Strict equivalence for genus 0

Suppose that X has genus 0. Then X has a rational point since $\mathbb{C}(z)$ is a C_1-field. Therefore $F = \mathbb{C}(z)(u)$ is the function field of X and $u' = g(u,z)$ for some $g(u,z) \in F$. Thus this equation is strictly equivalent to f.

It is easily verified that f has PP if and only if $g(u,z) = a_0(z) + a_1(z)u + a_2(z)u^2$. Indeed, PP is equivalent to the derivation D, given by $D(z) = 1$, $D(u) = g(u,z)$, having no poles.

Consider the equation $u' = u \cdot G(u,z)$ and $G(u,v) = \alpha_0 \prod (u - \alpha_j)^{n_j}$ with all α_* in a finite extension of $\mathbb{C}(z)$. This equation is strictly equivalent to an autonomous one, i.e., $v' = h(v)$ with $h(v) \in \mathbb{C}(v)$, if and only if $u = \frac{av+b}{cv+d}$ with a,b,c,d in a finite extension of $\mathbb{C}(z)$, $ad - bc \neq 0$ and

$$\frac{u'}{u} = \frac{a'v + b' + ah(v)}{av+b} - \frac{c'v + d' + ch(v)}{cv+d} = \alpha_0 \cdot \prod (\frac{av+b}{cv+d} - \alpha_j)^{n_j}.$$

From this equality one can make a guess for $av+b$ and/or $cv+d$ in case this term is not 1 and not a multiple of $z + \beta$ with $\beta \in \mathbb{C}$. This method may solve in some cases the question whether the equation is strictly equivalent to an autonomous

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equation.

The problem of deciding strict equivalence between two equations $u'_1 = g_1(u_1, z)$ and $u'_2 = g_2(u_2, z)$ seems to be, like the problem of finding algebraic solutions, 'recursive enumerable'. Indeed, one has to investigate whether for some algebraic A, B, C, D (with $AD - BC = 1$) the transformation $u_1 \mapsto u_2 := \frac{Au_1 + B}{Cu_1 + D}$ maps the first equation to the second. For fixed positive integers d, n, the Gröbner algorithm produces an answer for A, B, C, D with degree $\leq d$ over $\mathbb{C}(z)$ and such that the coefficients of their equations over $\mathbb{C}(z)$ are rational functions of degrees $\leq n$. Missing for a true algorithm is again an a priori bound on d, n. A special case of the problem is the following:
If the equation $u' = g(u, z)$ is strictly equivalent to the equation $v' = 0$ by a transformation $u = \frac{Av + B}{Cv + D}$ with A, B, C, D algebraic over $\mathbb{C}(z)$ and $AD - BC = 1$, then $u_c := \frac{Ac + B}{Cc + D}$ is a generic algebraic solution. In [A-C-F-G] it is shown that a generic algebraic solution exists if and only if the differential polynomial $F = u' - g(u, z)$ has zero remainder with respect to a certain standard differential polynomial depending upon a number of positive integers. It is remarked in [A-C-F-G] that there is no a priori bound known for these integers if the equation is not autonomous. This is in accordance with our opinion that there is no algorithm for the question whether equations like $y' = y^3 + z$ have algebraic solutions.

A heuristic indication that no algorithm for finding algebraic solutions exists is the order two equation $(\frac{zy'}{y})' = 0$ of low complexity. One observes namely that the algebraic solutions z^a with $a \in \mathbb{Q}$ have arbitrary complexity.

Remark 4.1 (*Properties of an autonomous equation of genus 0*). An autonomous equation with genus 0 has the form $v'k(v) = 1$ with $k(v) \in \mathbb{C}(v)^*$ (or is the trivial equation $v' = 0$). Write $k(v) = \sum_j \frac{a_j}{v - b_j} + \frac{d}{dv}(k_0(v))$, where the $a_j \notin \mathbb{Z}$ and $k_0(v) \in \mathbb{C}(v)$. By integration one finds a "functional equation" for the solutions, namely $\sum_j a_j \log(v - b_j) + k_0(v) = z + c$.

If the logarithmic terms are not present in this formula and the rational function $k_0(v)$ has degree 1, then the only moving singularities of solutions v are poles and the equation has PP.

If there are no logarithmic terms but $k_0(v)$ has degree > 1, then there are moving branch points for the solutions.

Suppose that $k_0(v) = 0$ and thus $z + c = \sum a_j \log(v - b_j)$. Now z has as function of v logarithmic singularities. One would expect that v has exponential singularities, like $e^{\frac{1}{z-a}}$. In the general case one expects branch points singularities and

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singularities of the type $e^{\frac{1}{\sqrt[m]{z-a}}}$ (a mixing of branching and exponential singularities).

Remark 4.2 (*Infinitesimal automorphism*). An *infinitesimal automorphism* of an order one equation f is, by definition, a $\mathbb{C}(z)$-linear derivation D of the field of fractions $F = \mathbb{C}(z)(s,t)$ of $\mathbb{C}(z)[s,t,\frac{1}{d}]$, commuting with the differentiation on F. We note that D is determined by $D(t) := h \in F$ and that h should satisfy the equation $h' = D(s)$ and $D(s)$ is given by the identity

$$0 = D(f(s,t)) = D(s) \cdot \frac{\partial f}{\partial s} + h \cdot \frac{\partial f}{\partial t}.$$

For the genus 0 case and $u' = g(u,z) \in F = \mathbb{C}(z)(u)$, the condition on $h := D(u)$ is $h\frac{\partial g}{\partial u} - g\frac{\partial h}{\partial u} = \frac{\partial h}{\partial z}$. For an autonomous equation of genus 0, i.e., $g \in \mathbb{C}(u)$, there exists a non trivial infinitesimal automorphism. Indeed, $D(u) = \lambda g$ with $\lambda \in \mathbb{C}^*$ obviously satisfies the above equation. A general equation f of genus 0 has no infinitesimal automorphisms. Indeed, a computation shows that the equation $u' = a_1 u + a_0$, with general $a_0, a_1 \in \mathbb{C}(z)$, has no infinitesimal automorphism $D \neq 0$.

5 Strict equivalence for genus 1

Let, for $i = 1,2$, the curves X_i with function fields $\mathbb{C}(z)(s_i, t_i)$ associated to the order one differential equation f_i, have genus 1. After a finite extension K of $\mathbb{C}(z)$, the curves have a point P_i. The classical method of using the meromorphic functions on X_i with only a pole at P_i yields $K(s_i, t_i) = K(x,y)$ with $y^2 = x^3 + a_i x + b_i$ and $a_i, b_i \in K$. The j-invariant classifies elliptic curves over an algebraically closed field. Hence a necessary condition for strict equivalence of f_1 and f_2 is equality of the j-invariants.

Suppose that the j-invariants coincide. Then after replacing K by a finite extension, we may identify the function fields of X_1 and X_2 with $K(x,y)$, $y^2 = x^3 + ax + b$. Let D_1, D_2 denote the two $\mathbb{C}$-linear derivations on $K(x,y)$, induced by f_1 and f_2. Then f_1 and f_2 are strictly equivalent if and only if there exists a $\overline{K}$-linear automorphism A of the field $\overline{K}(x,y)$ such that $D_2 = AD_1A^{-1}$. The group of the automorphisms $Aut(E)$ of the elliptic curve E over $\overline{K}$ corresponding to $\overline{K}(x,y)$, has a normal subgroup $E(\overline{K})$ of translations and $Aut(E)/E(\overline{K})$ is a cyclic group of order $2, 4$ or 6.

As in the case of first order equations of genus 0, the problem of strict equivalence is 'recursive enumerable' due to the large group $E(\overline{K})$. Missing for a true

algorithm is an a priori estimate on the degree of the field extension $\tilde{K}$ of $\mathbb{C}(z)$ and of the 'height' of the element in $E(\tilde{K})$ needed for a possible automorphism A.

Let the order one equation $f(y', y, z) = 0$ have genus 1. If the j-invariant is not in $\mathbb{C}$, then f is not strictly equivalent to a semi-autonomous equation.

In the other case, we may suppose that f corresponds, after a finite extension of K, to a differential field $K(x,y)$ with $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{C}$. The differentiation $'$ on the field $K(x,y)$ can be computed and is determined by $x' = a_0(x,z) + a_1(x,z)y$ (say with $a_1(x,z) \neq 0$). Thus we have replaced the original equation $f(y', y, z) = 0$ by $(\frac{x' - a_0(x,z)}{a_1(x,z)})^2 = x^3 + ax + b$. This differential equation is far from unique, since it depends on the choice of the 'origin' P of the elliptic curve. It seems not possible to decide whether the given equation f is strictly equivalent to an autonomous equation. However, the verification of PP does not depend on the particular choice of the point P (see [Mun-vdP] for details).

The difficulty in making strict equivalence explicit for the case that the curve associated to f has genus 0 or 1, is due to the large group of automorphisms of X. We will see that for hyperelliptic curves the situation is different.

6 Hyperelliptic curves

Let the pair (X, D), consisting of a curve X over a finite extension K of $\mathbb{C}(z)$ and of a $\mathbb{C}$-linear derivation D of the function field of X satisfying $D(z) = 1$, correspond to the order one equation $f(y', y, z) = 0$. We suppose that the genus g of X is ≥ 2.

An algorithm, due to J. Coates, computes for a curve given by an irreducible plane equation, an explicit basis of $H^0(X, L)$, where L is any line bundle. In particular, this algorithm computes an explicit basis of the g-dimensional vector space $H^0(X, \Omega_{X/K})$ over K (in fact for a number field K; the function field case is similar). For a closed point x of $\overline{K} \times_K X$, one chooses a local parameter t and considers the map $\overline{K}$-linear map $\ell_x : \overline{K} \otimes_K H^0(X, \Omega_{X/K}) \to \overline{K}$, given by $\ell_x(\omega) = a$ if, locally at x, one has $\omega = f dt$ and $f(x) = a$. A change of t has the effect of multiplying ℓ_x by an element in $\overline{K}^*$. This yields an algorithmic description of the canonical morphism $\phi : X \to \mathbb{P}(H^0(X, \Omega_{X/K})^*) \cong \mathbb{P}_K^{g-1}$.

For genus two, $\mathbb{P}(H^0(X, \Omega_{X/K})^*)$ is the projective line over K and we obtain an explicit degree two map $\phi : X \to \mathbb{P}_K^1$. This leads to an explicit equation

$y^2 = P(x)$, with $P(x) \in K(x)$, where x is a parameter for the projective line over K. Since the genus is two, one can take $P(x)$ to be a separable polynomial of degree six.

If the genus g is > 2, then for a 'general' curve X, the morphism ϕ is the canonical embedding of X into $\mathbb{P}_K^{g-1}$. The curves for which ϕ is not an embedding are called *hyperelliptic*. It is known that (see, e.g., [vL-vdG], in that case, the image of ϕ in $\mathbb{P}_K^{g-1}$ is a genus zero curve, called $(g-1)$-uple curve. Since the field K is a C_1-field, the genus zero curve is isomorphic to $\mathbb{P}_K^1$. The $(g-1)$-uple curve is an embedding $\mathbb{P}_K^1 \to \mathbb{P}_K^{g-1}$, given by $(x_0 : x_1) \in \mathbb{P}_K^1 \mapsto (x_0^{g-1} : x_0^{g-2}x_1, \ldots, : x_1^{g-1}) \in \mathbb{P}_K^{g-1}$, in suitable coordinates. The resulting morphism $X \to \mathbb{P}_K^1$ has degree two. The curve X over K is then represented by an explicit equation $y^2 = P(x)$ where $P(x) \in K[x]$ can be chosen to be a separable polynomial of degree $2g+2$.

The main observation is the existence of an algorithm computing an equation $y^2 = P(x)$, with separable $P(x) \in K[x]$ of degree $2g+2$, for a curve X over K of genus $g \geq 2$ which is known to be hyperelliptic. Moreover, the divisor of $P(x)$ in $\mathbb{P}_K^1$ is unique up to automorphisms of $\mathbb{P}_K^1$.

Testing (semi-)autonomous.

Let the pair (X,D) be derived from the order one equation $f(y',y,z) = 0$ and suppose that X is a hyperelliptic curve over K of genus $g \geq 2$. Let $y^2 = P(x)$ with $P(x) \in K(x)$ a separable polynomial of degree $2g+2$ and let R denotes its divisor. Suppose that $\overline{K} \times_K X$ is isomorphic to $\overline{K} \times_\mathbb{C} X_0$. Then, as above, one obtains an equation $y^2 = v$, with $v \in \mathbb{C}[x]$ a separable polynomial of degree $2g+2$, for X_0. Its divisor $\tilde{R}$ on $\mathbb{P}_\mathbb{C}^1$ is unique up to automorphisms of $\mathbb{P}_\mathbb{C}^1$.

The isomorphism between $\overline{K} \times_K X$ and $\overline{K} \times_\mathbb{C} X_0$ induces an isomorphism between the two projective lines $\mathbb{P}_{\overline{K}}^1$ and $\overline{K} \times \mathbb{P}_\mathbb{C}^1$ which sends the divisor R to $\tilde{R}$. We conclude the following.

Proposition 6.1. *The equation f is semi-autonomous (i.e., there is an isomorphism $\overline{K} \times_K X \to \overline{K} \times_\mathbb{C} X_0$) if and only if there exists an element $A \in \mathrm{PGL}_2(\overline{K})$ such that the divisor AR is defined over the subfield $\mathbb{C}$ of $\overline{K}$.*

It is easy to verify the existence of A in Proposition 6.1. After a finite extension of K, we may suppose that $P(x) = \prod_{r \in R}(x-r)$ where $R \subset K$ has cardinality $2g+2$. Then one defines A by, say, A maps three distinct elements r_1, r_2, r_3 of R to 1,2,3. Then f is semi-autonomous if and only if $A(R) \subset \mathbb{P}^1(\mathbb{C})$.

Suppose that f is semi-autonomous. Then one computes on the field $K(X)$,

which is identified with $K(y,x)$ with $y^2 = v \in \mathbb{C}[x]$ (as above), the action of the differentiation D. Then f is strictly equivalent to an autonomous equation if and only if $D(x) \in \mathbb{C}(x,y)$.

Testing strict equivalence.
An algorithm for testing strict equivalence between two equations f_1 and f_2 of genus $g \geq 2$ can be obtained in a similar way. After a finite extension K of $\mathbb{C}(z)$, the two fields are given for $j = 1, 2$ by equations $y^2 = \prod_{r \in R_j}(x - r)$ where R_1, R_2 are subsets of $\mathbb{P}^1(K)$ of cardinality $2g + 2$. The two fields are isomorphic if and only if some automorphism A of $\mathbb{P}^1_K$ which maps three chosen elements of R_1 to three chosen elements of R_2, has the property $A(R_1) = R_2$.

If A exists then we identify the two fields with the field corresponding to $y^2 = \prod_{r \in R}(x - r)$, where $R \subset \mathbb{P}^1(K)$ consists of $2g + 2$ elements. The automorphism group of this field is very explicit. It is generated by the involution $y \mapsto -y$, $x \mapsto x$ and the finite group of the automorphism of $\mathbb{P}^1_K$ preserving the set R.

Let D_1, D_2 denote the two derivations of this field coming from the equations f_1, f_2. Then f_1 and f_2 are strict equivalent if and only if there exists an automorphism A with $D_2 = AD_1A^{-1}$.

Example 6.2. The "standard" autonomous equation for a genus two curve is

$$(y')^2 - \prod_1^6 (y - a_j) = 0,$$

where the a_j are distinct elements of $\mathbb{C}$. The function field of the equation (over $\mathbb{C}(z)$) is $F = \mathbb{C}(z)(s,t)$, with equation $s^2 - \prod_1^6(t - a_j) = 0$, and the differentiation is given by $t' = s$ and $s' = \frac{1}{2}\frac{dP}{dt}$, where $P(t) = \prod_1^6(t - a_j)$.

One can put the standard equation in disguise by choosing a $T \in F$ such that $F = \mathbb{C}(z)(S,T)$, with $S = T'$.
(a). The choice $T = f \cdot t$ with $f \in \mathbb{C}(z)^*$ leads to the equations

$$f^2(fS - f'T)^2 - \prod_1^6 (T - fa_j) = 0.$$

For example, if we take $f = z^{-1}$ then the equation above becomes

$$(zS + T)^2 - \prod_1^6 (zT - a_j) = 0.$$

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(b). The choice $T = f \cdot s$ with $f \in \mathbb{C}(z)^*$ is also possible. First we consider the case $f = 1$. Then $\mathbb{C}(s, s') = \mathbb{C}(s, t)$. Indeed, we have $s^2 = P$, $\frac{dP}{dt} \in \mathbb{C}(s, s')$ and $\mathbb{C}(P, \frac{dP}{dt}) = \mathbb{C}(t)$.

Let $G \in \mathbb{C}[X_1, X_2]$ be the irreducible polynomial satisfying

$$G(s, s') = 0. \tag{$\star$}$$

Then this is another autonomous equation for our genus two field $\mathbb{C}(s, t)$.

Now the choice $T = f.s$ and $T' = S$ produces the order one differential equation

$$G\left(\frac{S - \frac{f'}{f}T}{f}, \frac{T}{f} \right) = 0.$$

We can make the equation $(\star)$ explicit as follows. There is a rational function $Q(X_1, X_2) \in \mathbb{C}(X_1, X_2)$ such that $t = Q(P, \frac{1}{2}\frac{dP}{dt})$. The the equation G between s' and s is obtained from $s^2 = P(t) = P(Q(s^2, s'))$.

As an example, consider $s^2 = P(t) = t^6 - 1$. Then we get $t = \frac{6P}{\frac{dP}{dt}}$, and therefore the equation $s^2 = t^6 - 1$ becomes

$$G(s, s') = s^2(s')^6 - 6(s^2)^6 + (s')^6 = 0.$$

7 Genus three and non hyperelliptic

Testing strict equivalence.
Suppose that the order one differential equations f_1, f_2 define the pairs (X_j, D_j), $j = 1, 2$ consisting of a genus 3 curve over K which is not hyperelliptic and a $\mathbb{C}$-linear derivation on the function field of this curve, satisfying $D_j(z) = 1$.

First one wants to investigate whether the curves become isomorphic after a finite extension of K. The curve X_j has a canonical embedding as a smooth curve in $\mathbb{P}^2_K$, given by a homogeneous polynomial F_j of degree 4 (unique up to constants). Further $\overline{K} \times_K X_1$ is isomorphic to $\overline{K} \times_K X_2$ if and only if there exists an automorphism A of $\mathbb{P}^2_{\overline{K}}$ such that $F_2 = AF_1$ (up to constants).

Let Z denote the variety of the homogeneous polynomials of degree 4 (up to multiplication by constants and defining a smooth curve). On this variety the group PGL_3 acts in a natural way. The naive quotient Z/PGL_3 does not exist.

However, one can compute generators $\{I_k\}$ for the ring of the PGL_3-invariant homogeneous functions on Z. This defines a 'coarse moduli space'. For a base field which is algebraically closed (in our case $\overline{K}$), two degree 4 homogeneous polynomials F_1, F_2 are equivalent (up to constants) under PGL_3 if and only if the basic invariants $\{I_k\}$ have the same values in F_1 and F_2. An explicit computation of the $\{I_k\}$ is given in [R]. Moreover, this thesis contains an algorithm which produces A with $F_2 = AF_1$ (up to constants), whenever $I_k(F_1) = I_k(F_2)$ for all k.

Another interesting method testing whether X_1, X_2 become isomorphic over an extension of K can be deduced from [F], Proposition 1.1. The idea is that one provides the smooth degree 4 curves with an additional structure such that the new space Z^+, consisting of these curves with extra structure, admits a good quotient by PSL_3 (which is a 'fine moduli space' for the problem considered here).

This additional structure consists, for a degree 4 smooth curve $X \subset \mathbb{P}^2$, of two bitangents L_1, L_2 and the tangent points A_1, B_1 on L_1 and the tangent points A_2, B_2 on L_2. The map $Z^+ \to Z$, which forgets the extra structure, is finite surjective and has degree $28 \times 27 \times 2 \times 2$, since a smooth degree 4 curve has 28 bitangents.

Now computing a possible isomorphism $\overline{K} \times_K X_1 \to \overline{K} \times_K X_2$ can be done as follows. Let F_1 be the equation of the embedded X_1 and (after an extension of K) we choose $(L_1, L_2, A_1, B_1, A_2, B_2)$. Let F_2 be the equation for X_2 and consider any of the possible tuples $(L_1^*, L_2^*, A_1^*, B_1^*, A_2^*, B_2^*)$ for F_2. Let $\phi \in \mathrm{PGL}_3$ be the unique transformation ϕ which $(A_1, B_1, A_2, B_2) \mapsto (A_1^*, B_1^*, A_2^*, B_2^*)$. One computes whether $\phi F_1 = F_2$ (up to scalars). If this has no success for any of the possible tuples, then $\overline{K} \times_K X_1$ is not isomorphic to $\overline{K} \times_K X_2$.

Suppose now that X_1 can be identified with X_2 (after replacing K by a finite extension). Then for strict equivalence one has to test whether $D_2 = AD_1A^{-1}$ holds for some element A in the known finite group of automorphisms of $X_1 = X_2$ (again possibly extending K).

Testing strict equivalence to a (semi-)autonomous equation.
Let (X, D) denote a curve of genus 3 over K which is not hyperelliptic and D a derivation of the function field of X such that $D(z) = 1$. As before, X yields a homogeneous polynomial F of degree 4. The curve is semi-autonomous if and only if the values of the invariants I_k for F are in $\mathbb{C}$. If X is semi-autonomous, then according to [R], there is an algorithm producing $A \in \mathrm{PGL}_3(\overline{K})$ such that $F_0 := A(F)$ has its coordinates in $\mathbb{C}$. The homogeneous polynomial F_0 of degree 4 defines a curve X_0 over $\mathbb{C}$ such that $\overline{K} \times_K X \cong \overline{K} \times_{\mathbb{C}} X_0$. Then (X, D) is strictly equivalent to an autonomous equation if and only if D leaves the function field of

X_0 invariant.

8 Non hyperelliptic curves of higher genus

Let, as before, the pair (X,D) correspond to an order one differential equation. Suppose that X/K has genus $g \geq 3$ and that X is not hyperelliptic. Testing strict equivalence and equivalence to a (semi-)autonomous equation can be done as in §6 if one has a reasonable explicit (coarse or fine) moduli space for non hyperelliptic curves of genus g and a way of determining the finite group of automorphisms of a given curve of this type.

The following method shows the existence of an algorithm based upon properties of the Weierstrass points of a curve. Let X_j, $j = 1,2$ denote non hyperelliptic curves of genus $g \geq 3$ over K. The canonical embeddings $X_j \subset \mathbb{P}_K^{g-1}$ are explicit. Let $W_j \subset \overline{K} \times_K X_j$ denote the finite (and effectively computable) set of Weierstrass points. After a finite extension of K we may suppose that the points of W_1, W_2 are K-rational. An isomorphism $\phi : \overline{K} \times X_1 \to \overline{K} \times X_2$ is induced by a unique automorphism ψ of $\mathbb{P}_K^{g-1}$ which maps W_1 to W_2. There are only finitely many automorphisms ψ such that $\psi(W_1) = W_2$. One can test these ψ's for the properties $\psi(X_1) = X_2$ and $\psi^{*-1}D_2\psi^* = D_1$. This yields an algorithm as desired.

Acknowledgements

References

[A-C-G-H] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris. *Geometry of Algebraic Curves, Volume 1.* Grundlehren der Math. Wiss. 267, Springer-Verlag, Berlin, 1985.

[A-C-F-G]   J.M. Aroca, J. Cano, R. Feng, X-S. Gao. Algebraic General Solutions of Algebraic Ordinary Differential Equations. *ISSAC '05.* 29-36, ACM, New York, 2005.

[Bro]   M. Bronstein. Integration of elementary functions. *J. Symbolic Comput.* **9** (1990), 117-173.

[Co]   J. Coates, *Construction of rational functions on a curve*. Proc. Cambridge Philos. Soc., **68** (1970) 105123.

[F]   C. Faber. *Chow rings of moduli spaces of curves I: Yhe Chow ring of* $\overline{M}_3$. Ann. of Math. **132** (1990), 331-419.

[K]   E.R. Kolchin. *Differential algebra and algebraic groups.* Academic Press, 1973.

[Mun-vdP] G. Muntingh, M. van der Put. Order one equations with the Painlevé property. *Indag. Math. (N.S.)* **18** (2007), 83-95.

[vL-vdG]   J. H. van Lint, G. van der Geer. *Introduction to Coding Theory and Algebraic Geometry.* Birkhäuser Verlag, 1989.

[vdP-S]   M. van der Put, M. F. Singer. *Galois theory of linear differential equations.* Grundlehren der Math. Wiss. 328, Springer-Verlag, Berlin, 2003.

[Ri]   J.F. Ritt. *Differential Algebra.* A.M.S.Colloquium Publications 33, 1950.

[R]   S. van Rijnswou. *Testing the Equivalence of Planar Curves.* PhD Thesis, Eindhoven University of Technology, 2001 – ISBN 90-386-0861-6.

[Tra]   B. M. Träger. *Integration of Algebraic Functions.* Ph.D. Thesis, Massachusetts Institute of Technology, 1984.