International Journal of Biomathematics
(2025) 2550146 (15 pages)

DOI: 10.1142/S1793524525501463

On the existence of forced extinction wave and spreading speed for delayed population in time-periodic deteriorated environment

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Received 6 June 2025
Revised 21 October 2025
Accepted 11 November 2025
Published 1 December 2025

Communicated by Meng Fan

In this communication, we are concerned with the propagation phenomena of a delayed equation without quasimonotonicity modeling of the dynamics of a population in a time-periodic environment, in which is assumed that temporally-average shifts from a favorable zone to an unfavorable zone. First, by employing the technique in [J. S. Guo, A. A. L. Poh and C. C. Wu, Forced waves of saturation type for Fisher-KPP equation in a shifting environment, *Appl. Math. Lett.* **140** (2023) 108573], we prove the existence of periodic forced wave as $c > c^*$ in the spirit of [G. Lin, Spreading speeds and traveling wave solutions for a delayed periodic equation without quasimonotonicity, *J. Dyn. Differ. Equations* **31**(4) (2019) 2275–2292], employing Schauders fixed point theorem and regularity of analytic semigroups by tactfully constructing a pair of generalized super- and sub-solutions without assuming monotonicity on the initial growth rate of population. Our result confirms that the delayed death rate in our model does not prevent the occurrence of forced extinction waves in deteriorated environment. Furthermore, by providing suitable auxiliary equations, we derive the spreading speed for this model. Our work is a counterpart of the interesting results obtained in [J. Fang, R. Peng and X. Q. Zhao, Propagation dynamics of a reaction–diffusion equation in a time-periodic

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shifting environment, *J. Math. Pures Appl.* **147** (2021) 1–28; J. S. Guo, A. A. L. Poh and C. C. Wu, Forced waves of saturation type for Fisher-KPP equation in a shifting environment, *Appl. Math. Lett.* **140** (2023) 108573; G. Lin, Spreading speeds and traveling wave solutions for a delayed periodic equation without quasimonotonicity, *J. Dyn. Differ. Equations* **31**(4) (2019) 2275–2292].

Keywords: Forced extinction wave; spreading speed; delayed population; time-periodic shifting environment.

1. Introduction

The study of the impact of climate change on the dynamics of a biological species is one of the challenging topics in contemporary applied mathematics. This communication is concerned with the propagation phenomena of a delayed reaction–diffusion equation with spatial diffusion in a time-periodic shifting environment, which reads as follows

$$\begin{cases} u_t(t,x) = du_{xx}(t,x) + u(t,x)[\rho(t,x-ct) - g(t)u(t,x) - h(t)u(t-\tau,x)], \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ \ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $t > 0$, $x \in \mathbb{R}$, $d > 0$ is a diffusion constant, $c > 0$ is a shifting speed, $g(t) > 0$, $h(t) \geq 0$ are T-periodic functions for some $T > 0$, $\tau \in (0, T)$ denotes the time delay and $\rho = \rho(t, s)$ is an environment shifting function that is T-periodic in t.

In a pioneering work, Berestycki *et al.* [4] first proposed the models of a heterogeneous reaction–diffusion equation with a forced speed $c > 0$ as

$$u_t = u_{xx} + f(x - ct, u) \quad x \in \mathbb{R}, \ \ t > 0,$$

to study the impact of climate change on the dynamics of population in a shifting environment. In [4], the authors studied the case that the favorable habitat of species is compactly supported for a single species. Next, Berestycki *et al.* [3] considered a situation where spatial heterogeneity leads to a cline, a gradual transition in dominance of a system of competing species. Later, Vo *et al.* [12, 24] also obtained existence, uniqueness and global stability of more general models than [4] in higher dimensional space but reducing the condition that the favorable zone is not necessarily compactly supported. On the other hand, Li *et al.* [14] and Fang *et al.* [6] investigated the propagation dynamics of species under various climate change conditions. More precisely, the environment in [14] is assumed to be locally shifting from favorable to unfavorable while in [6], the authors considered the reverse case assuming that the environment is shifting from unfavorable to favorable. Then, in a deep work, Fang *et al.* [7] studied the equation with time periodic asymptotically KPP type as $s \to -\infty$ as follows:

$$\begin{cases} u_t = u_{xx} + ug(t, x - ct, u), \\ u(0, x) = u_0(x). \end{cases}$$

Under a sub-homogeneity condition, the authors [7] proved the existence and uniqueness of the periodic forced wave if and only if $|c| < c^* :=$

$2\sqrt{\frac{1}{T}\int_0^T g(t,-\infty,0)dt} > 0$ and showed that it attracts parabolic solution depending on the tail behavior of the initial value. We also refer to Wang *et al.* [26] and references therein for the studies of time-periodic reaction–diffusion systems related to the current framework. Besides, delayed reaction–diffusion equations are also the subject of intensive research for a long time. The propagation dynamics for delayed equations has been strongly studied, for instance, the readers can refer to the interesting works of Lin [17], Solar and Trofimchuk [23], Wang *et al.* [25], Wu and Xu [27], Zhang and Lin [29] and references for the frameworks and ideas related to the current investigation. In particular, Lin [17] investigated a periodic delayed model with spatial diffusion

$$\begin{cases} u_t(t,x) = du_{xx}(t,x) + u(t,x)[r(t) - a(t)u(t,x) - b(t)u(t-\tau,x)], \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \end{cases}$$

where $t > 0$, $x \in \mathbb{R}$, $d > 0$ is a diffusion constant, $a(t) > 0$, $b(t) \geq 0$, $r(t)$ are T-periodic functions and $\overline{r} := \frac{1}{T}\int_0^T r(s)ds > 0$. The author [17] studied the existence, nonexistence and spreading speed of periodic forced wave solutions by employing the Schauder's fixed point theorem, the regularity of analytic semigroup, generalized super- and sub-solutions and some auxiliary equations, some interesting works related to the global dynamics of delayed reaction–diffusion equations for epidemic models can be quoted by Sun and coauthors [22, 28]. Very recently, Lv *et al.* [20] also obtained the existence and uniqueness of forced wave in time-delayed nonlocal dispersal equations, with different form of nonlinearity, in shifting habitats using the monotone iteration and the sliding technique. The authors [20] further showed that time delay does not prevent the occurrence of forced extinction waves for non-locally diffusive populations in deteriorated environment. We also emphasize that the impact of climate change on population dynamics has been extensively investigated from different aspects, for instance, on vegetation patterns by Li *et al.* [15] and under free boundary setting by Du and coauthors [2, 5, 10, 13] and Hu *et al.* [11].

It is natural to incorporate the impact of climate change in the study of dynamics for delayed populations proposed by Lin [17], which is one of the important topics in the contemporary applied mathematics. In this work, we investigate the spreading speed of Eq. (1.1) for a population initially occupying a compact subset of $\mathbb{R}$ and establish an existence criterion for the forced extinction wave described by

$$\begin{cases} U_t = dU_{\xi\xi} + cU_\xi + U[\rho(t,\xi) - g(t)U - h(t)U(t-\tau, \xi + c\tau)], \\ U(t,\xi) = U(t+T,\xi), \end{cases} \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}, \tag{1.2}$$

together with

$$\lim_{\xi \to -\infty} U(t,\xi) = \lim_{\xi \to \infty} U(t,\xi) = 0 \quad \text{uniformly in } t > 0, \tag{1.3}$$

where we set $\xi = x - ct$ and $U(t,\xi) := u(t,x)$. This wave represents a moving stable state of the species in the long run. On the biological aspect, it describes that the extinction occurs when the species move either too rapidly or too slowly relative to the shifting speed of the environment.

For any given $X \subset \mathbb{R}^2$, we denote by $\mathcal{BUC}(X,\mathbb{R})$ the space of bounded and uniformly continuous functions from X into $\mathbb{R}$. For $\beta \in (0,1)$, we consider $\mathcal{BC}^{\beta}(\mathbb{R},\mathbb{R})$ is the space of β-Hölder continuous functions on $\mathbb{R}$ and $\mathcal{BC}^{\beta,2\beta}([a,b]\times\mathbb{R},\mathbb{R})$ is the space of β-Hölder continuous and 2β-Hölder continuous functions on $[a,b]\times\mathbb{R}$. Throughout this whole paper, we assume the following hypotheses.

Hypothesis (H1): The function $\rho = \rho(t,s) \in \mathcal{BC}^{\theta}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ for some $\theta \in (0,1)$ and ρ is a T-periodic function in $t \in \mathbb{R}$.

The function $\rho(t, x-ct)$ with $c > 0$ expresses the impact of climate change on the initial growth rate of population via the shifting environment. The function $\rho(t, x-ct)$ thus divides the spatial domain into two parts: the region with good-quality habitat suitable for growth (i.e. $\rho(t,x-ct) > 0$), and the region with poor-quality habitat unsuitable for growth (i.e. $\rho(t,x-ct) < 0$) and the time-periodicity expresses seasonal influence on the environment, for instance, the temperature and rainfall are expected to be almost similar in the same period every year. Note that we do not *a priori* assume that the function $\rho(t,s)$ is monotone with respect to s, see also [8, 9].

Hypothesis (H2): Assume the two limits $\alpha_{\pm}(t) := \lim_{s\to\pm\infty}\rho(t,s)$ exist uniformly in $t\in\mathbb{R}$. Put $\overline{\alpha}_{\pm} = \frac{1}{T}\int_0^T \alpha_{\pm}(t)dt$. We further assume $-\infty < \overline{\alpha}_- < 0 < \overline{\alpha}_+ < \infty$.

This condition implies that the temporal medium of the environment is unfavorable at $-\infty$ and favorable at ∞ in the spirit of Fang *et al.* [7]. However, our hypothesis is the reverse of the condition (G3) and (G4) used in [7], which expresses that the temporal medium of environment is locally shifting from favorable to unfavorable in the long run. This leads us to an opposite phenomenon that the forced wave exists if $c > c^*$ while in [7] the forced wave holds as $c < c^*$.

Hypothesis (H3): $g = g(t) > 0$, $h = h(t) \geq 0$ are T-periodic and in $\mathcal{BC}^{\theta}(\mathbb{R},\mathbb{R})$.

For the sake of convenience, we define $c^* := 2\sqrt{d\overline{\alpha}_+}$ and $\alpha(t) := \alpha_+(t)$ for $t \geq 0$. Since $\overline{\alpha} := \frac{1}{T}\int_0^T \alpha(t)dt > 0$, the following ODE $\frac{du}{dt} = u[\alpha(t) - g(t)u]$ admits a unique T-periodic asymptotically stable solution $u^* = u^*(t) > 0$ with positive initial value (see [16, 17] and reference therein).

Theorem 1. *Assume* (H1) *to* (H3) *are fulfilled. If* $c > c^*$, *then system* (1.2) *admits a positive periodic forced wave solution* U *satisfying*

$$\lim_{\xi\to\infty} U(t,\xi)e^{k\xi} = p(t) \quad \textit{uniformly in } t \in [0,T].$$

Suppose additionally that $\xi \mapsto \rho(t,\xi)$ *is non-decreasing, then*

$$\lim_{\xi\to-\infty} U(t,\xi) = 0 \quad \textit{uniformly in } t \in [0,T].$$

Furthermore, the solution of (1.1) *starting from* $u_0(0,\cdot) \in C_c(\mathbb{R})$, $u_0(0,\cdot) > 0$ *has the following spreading properties*:

(i) $\lim_{t\to\infty} \sup_{|x|>(c^*+\varepsilon)t} u(t,x) = 0$ *for any* $\varepsilon > 0$;
(ii) $\liminf_{t\to\infty} \inf_{(c+\varepsilon)t\le x\le(c^*-\varepsilon)t} u(t,x) > 0$ *for any* $0 < c < c^*$ *and* $\varepsilon \in (0, \frac{c^*-c}{2})$.

The constant k and the function $p(t)$ will be defined in Sec. 2.1. This type of spreading speed result has been investigated in Ahn *et al.* [1] for a climate change model. The regularity of semigroup theory and linear partial differential equations in [26] are invoked to prove the existence of periodic wave solution. We emphasize that to prove the existence of forced wave, we do not necessarily assume that the function $\rho(t,\cdot)$ is monotone with respect to the second variable. Our existence result is consistent with the existence of forced wave obtained by Lin [17], namely as $c > c^* > 0$ and opposite to the existence in Fang *et al.* [7] holding as $c < c^*$. On the biological interpretation, our result implies that, in the new model for delayed population under the environment locally shifting from favorable to unfavorable and the death rate is additionally contributed by the lagged individuals, then species can be persistent if they move fast enough. In the same spirit of [20] and Fang *et al.* [7, Theorem 1.2], our result confirms that the delayed death rate $h(t)u(t-\tau,x)$ does not prevent the occurrence of forced extinction waves in deteriorated environment. Furthermore, if the initial growth rate of population is spatially non-decreasing, we can characterize the spreading properties of forced wave solutions of (1.1) when the initial condition is compactly supported.

2. Proof of Main Results

Before proving the main results, let us recall the semigroup theory for (1.2) introduced in [17, 26]. Let $K > 0$ be a large enough constant such that

$$u \mapsto Ku - h(t)\left[\max_{t\in[0,T]} u^*(t)\right] u + u[\rho(t,\xi) - g(t)u];$$

$$u \mapsto Ku - h(t)\left[\max_{t\in[0,T]} u^*(t)\right] u + u[\alpha(t) - g(t)u]$$

are increasing in $u \in [0, \max_{t\in[0,T]} u^*(t)]$, for any $t \in [0,T]$, $\xi \in \mathbb{R}$. Consider the following differential operator $\mathcal{A} := d\frac{\partial^2}{\partial^2\xi} + c\frac{\partial}{\partial\xi} - K$ in a suitable domain $D(\mathcal{A})$. Then,

$$\begin{cases} \dfrac{du}{dt} = \mathcal{A}u, \\ u(0) = u_0 \in \mathcal{BUC}(\mathbb{R},\mathbb{R}) \end{cases}$$

admits an analytic semigroup $\mathcal{S}(t) : \mathcal{BUC}(\mathbb{R},\mathbb{R}) \to \mathcal{BUC}(\mathbb{R},\mathbb{R})$ (see [19, Sec. 2 and Subsec. 5.1.1]). Since $K > 0$, one has

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} [\mathcal{S}(t)u_0](\xi) = 0. \tag{2.1}$$

Next, we define

$$f(t,\xi,U,V) := KU(t,\xi) + U(t,\xi)[\rho(t,\xi) - g(t)U(t,\xi) - h(t)V(t-\tau,\xi+c\tau)],$$

for any given $t \geq 0$, $\xi \in \mathbb{R}$ and $U(t,\cdot)$, $V(t-\tau,\cdot) \in \mathcal{BUC}(\mathbb{R},\mathbb{R})$. Then,

$$\begin{cases} U(t,\xi) = [\mathcal{S}(t)U_0(0,\cdot)](\xi) + \displaystyle\int_0^t \mathcal{S}(t-s)f(t,\cdot,U,U)ds(\xi), \quad t \geq 0, \;\; \xi \in \mathbb{R}, \\ U(s,\xi) = \psi_0(s,\xi), \quad s \in [-\tau,0], \;\; \xi \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $\psi_0 \in \mathcal{BUC}([-\tau,0]\times\mathbb{R},\mathbb{R})$ is an initial condition. This formula defines a mild solution of the following initial value problem

$$\begin{cases} \dfrac{\partial U}{\partial t}(t,\xi) = \mathcal{A}U(t,\xi) + f(t,\xi,U,U), \quad t \geq 0, \;\; \xi \in \mathbb{R}, \\ U(s,\xi) = \psi_0(s,\xi), \quad s \in [-\tau,0], \;\; \xi \in \mathbb{R}. \end{cases} \tag{2.3}$$

We refer the reader to [18, Sec. 5], [19, Sec. 5] and [21] for details on semigroup theory, regularity and *a priori* estimations. To continue our work, we further consider

$$\mathcal{H}(t,s,\xi,U,V) := [\mathcal{S}(t)\psi_0(0,\cdot)](\xi) + \int_s^t \mathcal{S}(t-k)f(k,\cdot,U,V)dk(\xi),$$

where $s \in [0,t]$ and $U(s,\cdot)$, $V(s-\tau,\cdot) \in \mathcal{BUC}(\mathbb{R},\mathbb{R})$. With the notation, the generalized sub- and super-solutions are defined as follows:

Definition 2. For any given $0 < T' \leq \infty$, let $\underline{U}$ and $\overline{U}$ be two functions in $\mathcal{BC}([-\tau,T')\times\mathbb{R},\mathbb{R})$ satisfying

$$0 \leq \underline{U}(t,\xi) \leq \overline{U}(t,\xi) \leq u^*(t), \quad \forall\,(t,\xi) \in [-\tau,0]\times\mathbb{R}$$

and

$$\begin{aligned} &\overline{U}(t,\xi) \geq \mathcal{H}(t,s,\xi,\overline{U},\underline{U}), \\ &\underline{U}(t,\xi) \leq \mathcal{H}(t,s,\xi,\underline{U},\overline{U}), \quad \forall\, t,s \in [0,T'), \;\; \xi \in \mathbb{R}, \;\; s \leq t. \end{aligned}$$

We call ($\underline{U}$, $\overline{U}$) a pair of generalized sub- and super-solutions of (2.2).

By arguments similar to those in [17, Lemma 3.2], the following comparison and existence results hold:

Lemma 3. *Assume that* $(\underline{U},\overline{U}) \in \mathcal{BC}([-\tau,T')\times\mathbb{R},\mathbb{R})$ *is a pair of generalized sub- and super-solutions of* (2.2)*. Then, the following statements hold*:

(i) $\underline{U}(t,\xi) \leq \overline{U}(t,\xi)$, $\xi \in \mathbb{R}$, $t \in [0,T')$.
(ii) *If* $U_0 \in \mathcal{BUC}([-\tau,0]\times\mathbb{R},\mathbb{R})$ *satisfies* $\underline{U}(t,\xi) \leq U_0(t,\xi) \leq \overline{U}(t,\xi), \forall\, t \in [-\tau,0]$, $\xi \in \mathbb{R}$, *then* (2.2) *admits a unique solution* U *such that* $\underline{U}(t,\xi) \leq U(t,\xi) \leq \overline{U}(t,\xi), \forall\, t \in [0,T'), \xi \in \mathbb{R}$.

Let us introduce the functional space $\mathcal{B}_\mu(\mathbb{R})$ and the time-dependent version of them as follows:

$$|u|_\mu := \sup_{x\in\mathbb{R}} e^{-\mu|x|}|u(x)|, \quad \|u\|_\mu := \sup_{t\in[a,b],x\in\mathbb{R}} e^{-\mu|x|}|u(t,x)|,$$

$$\mathcal{B}_\mu(\mathbb{R}) := \{u \in \mathcal{BUC}(\mathbb{R},\mathbb{R}) : |u|_\mu < \infty\},$$

$$\overline{\mathcal{B}}_\mu(\mathbb{R}) := \{u \in \mathcal{BUC}([a,b]\times\mathbb{R},\mathbb{R}) : \|u\|_\mu < \infty\},$$

where $a, b \in \mathbb{R}$ and $\mu > 0$. One readily verifies that $\mathcal{B}_\mu(\mathbb{R})$ and $\overline{\mathcal{B}}_\mu(\mathbb{R})$ are Banach spaces.

Lemma 4. *Let A be a subset of $\mathcal{B}_\mu(\mathbb{R})$. Suppose for any given compact subset $K \subset \mathbb{R}$, A is precompact in $C(K,\mathbb{R})$ with supremum norm. Then, A is precompact in $\mathcal{B}_\mu(\mathbb{R})$.*

We omit the proof, as it follows from the technique in [26, Lemma 2.8]. Next, we construct a suitable pair of sub- and super-solutions for the equation.

2.1. *Sub- and super-solutions*

Inspired by the work [7], for $c > c^*$, let η be a small positive number and $k > 0$ be a solution of

$$dk^2 - ck + \overline{\alpha} - \eta = 0. \tag{2.4}$$

We choose k as the smaller root, which remains positive provided that $\eta > 0$ is sufficiently small. Set

$$p(t) := e^{\int_0^t [dk^2 - ck + \alpha(s) - \eta]ds} = e^{\int_0^t [\alpha(s) - \overline{\alpha}]ds}.$$

Then, p is T-periodic. Let $\epsilon > 0$ be small enough such that

$$k_\epsilon := k + \epsilon < 2k \quad \text{and} \quad d(k_\epsilon)^2 - ck_\epsilon + \overline{\alpha} - \eta < 0.$$

For t, $\xi \in \mathbb{R}$, we consider

$$\overline{U}(t,\xi) := \min\{p(t)e^{-k\xi}, u^*(t)\}; \quad \underline{U}(t,\xi) := \max\{p(t)(e^{-k\xi} - Le^{-k_\epsilon\xi}), 0\},$$

where $L > 0$ will be chosen large enough. Thanks to (H2), there exists $\xi(\eta) > 0$ such that

$$\rho(t,\xi) \geq \alpha(t) - \eta, \quad t \in \mathbb{R}, \ \forall\, \xi > \xi(\eta).$$

Next, we claim $(\underline{U}, \overline{U})$ is a pair of generalized sub- and super-solution.

Case 1: $\overline{U}(t,\xi) = u^*(t)$. One can verify that $\overline{U}$ is the super-solution.

Case 2: $\overline{U}(t,\xi) = p(t)e^{-k\xi} < u^*(t)$. Then, it follows from $\alpha = \alpha_+ \geq \rho$ and (2.4) that

$$\overline{U}_t - d\overline{U}_{\xi\xi} - c\overline{U}_\xi = \overline{U}(\alpha(t) - \overline{\alpha}) - dk^2\overline{U} + ck\overline{U}$$

$$= \overline{U}(\alpha(t) - \eta)$$
$$\geq \overline{U}[\rho(t,\xi) - g(t)\overline{U} - h(t)\underline{U}(t-\tau, \xi + c\tau)],$$

provided that η is small enough such that $g(t)\overline{U} > \eta$ for any $t \in [0,T]$. This is possible since $g(t)\overline{U} > 0$ on $[0,T]$.

Case 3: $\underline{U}(t,\xi) = 0$. It is easy to check that $\underline{U}$ is the sub-solution.

Case 4: $\underline{U}(t,\xi) = p(t)(e^{-k\xi} - Le^{-k_\epsilon \xi}) > 0$. It means that $\xi > \frac{\ln(L)}{\epsilon}$.

$$\begin{aligned}
&\underline{U}_t - d\underline{U}_{\xi\xi} - c\underline{U}_\xi \\
&\quad = p(t)e^{-k\xi}(\alpha(t) - \overline{\alpha} - dk^2 + ck) - Lp(t)e^{-k_\epsilon \xi}(\alpha(t) - \overline{\alpha} - dk_\epsilon^2 + ck_\epsilon) \\
&\quad = p(t)e^{-k\xi}(\alpha(t) - \eta) - Lp(t)e^{-k_\epsilon \xi}(\alpha(t) - \eta + \eta - \overline{\alpha} - dk_\epsilon^2 + ck_\epsilon) \\
&\quad = p(t)e^{-k\xi}(\alpha(t) - \eta) + Lp(t)e^{-k_\epsilon \xi}(dk_\epsilon^2 - ck_\epsilon + \overline{\alpha} - \eta) - Lp(t)e^{-k_\epsilon \xi}(\alpha(t) - \eta).
\end{aligned}$$

We choose $L > 0$ large enough so that

$$\frac{\ln(L)}{\epsilon} > \xi(\eta) \quad \text{or} \quad L > e^{\epsilon \xi(\eta)}.$$

Thus, one has

$$\begin{aligned}
&-\underline{U}[\rho(t,\xi) - g(t)\underline{U} - h(t)\overline{U}(t-\tau, \xi + c\tau)] \\
&\quad \leq -\underline{U}[(\alpha(t) - \eta) - g(t)\underline{U} - h(t)\overline{U}(t-\tau, \xi + c\tau)] \\
&\quad = -(\alpha(t) - \eta)p(t)e^{-k\xi} + L(\alpha(t) - \eta)p(t)e^{-k_\epsilon \xi} + g(t)\underline{U}^2 + h(t)\underline{U}\overline{U}(t-\tau, \xi + c\tau) \\
&\quad \leq -(\alpha(t) - \eta)p(t)e^{-k\xi} + L(\alpha(t) - \eta)p(t)e^{-k_\epsilon \xi} + g(t)p^2(t)e^{-2k\xi} \\
&\qquad + h(t)p(t)e^{-k\xi}p(t-\tau)e^{-k\xi - kc\tau\xi} \\
&\quad \leq -(\alpha(t) - \eta)p(t)e^{-k\xi} + L(\alpha(t) - \eta)p(t)e^{-k_\epsilon \xi} + g(t)p^2(t)e^{-2k\xi} \\
&\qquad + h(t)p(t)p(t-\tau)e^{-2k\xi}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\underline{U}_t - d\underline{U}_{\xi\xi} - c\underline{U}_\xi - \underline{U}[\rho(t,\xi) - g(t)\underline{U} - h(t)\overline{U}(t-\tau, \xi + c\tau)] \\
&\quad \leq e^{-k_\epsilon \xi}Lp(t)(dk_\epsilon^2 - ck_\epsilon + \overline{\alpha} - \eta) + g(t)p^2(t)e^{-2k\xi} + h(t)p(t)p(t-\tau)e^{-2k\xi} \\
&\quad = e^{-k\xi - \epsilon\xi}p(t)[L(dk_\epsilon^2 - ck_\epsilon + \overline{\alpha} - \eta) + g(t)p(t)e^{-k\xi + \epsilon\xi} + h(t)p(t-\tau)e^{-k\xi + \epsilon\xi}].
\end{aligned}$$

It is known that $e^{-k\xi - \epsilon\xi}p(t) > 0$. If $L > 1$, then $\xi > \frac{\ln(L)}{\epsilon} > 0$ leading to $e^{-k\xi + \epsilon\xi} < 1$. Hence, we can choose L large enough so that

$$L \geq -\frac{\sup_{t\in[0,T]}[g(t)p(t) + h(t)p(t-\tau)]}{dk_\epsilon^2 - ck_\epsilon + \overline{\alpha} - \eta} + e^{\epsilon\xi(\eta)} + 1 > 1,$$

and $\underline{U} \leq \overline{U}$ to obtain the desired result.

2.2. *Forced wave solutions*

We now state the existence of forced wave solutions.

Theorem 5. *Assume* (H1) *to* (H3) *hold. Let* $(\overline{U}, \underline{U})$ *be a pair of generalized super- and sub-solutions and* $\underline{U} \not\equiv 0$. *Suppose further that* $(\overline{U}(t,\xi), \underline{U}(t,\xi)) = (\overline{U}(t+T,\xi), \underline{U}(t+T,\xi))$, *for any* $t, \xi \in \mathbb{R}$. *Then,* (1.2) *admits a positive* T*-periodic solution* U *such that*

$$\underline{U}(t,\xi) \le U(t,\xi) \le \overline{U}(t,\xi) \quad \text{for any } t \in \mathbb{R} \text{ and } \xi \in \mathbb{R}.$$

Proof. Let $\phi = \phi(t,\cdot) \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ satisfy

$$\underline{U}(t,\xi) \le \phi(t,\xi) \le \overline{U}(t,\xi), \quad \phi(t,\xi) = \phi(t+T,\xi), \quad \forall\,(t,\xi) \in \mathbb{R}\times\mathbb{R}.$$

We consider the following integral equation

$$V(t,\xi) = [\mathcal{S}(t)V(0,\cdot)](\xi) + \int_0^t \mathcal{S}(t-s)f(t,\cdot,\phi,\phi)ds(\xi), \quad \forall\, t \in [0,T],\ \ \xi \in \mathbb{R}, \quad (2.5)$$

where $V(0,\xi)$ satisfies

$$\underline{U}(0,\xi) \le V(0,\xi) \le \overline{U}(0,\xi), \quad \forall\, \xi \in \mathbb{R}.$$

Therefore, thanks to the comparison principle (Lemma 3), Eq. (2.5) admits a unique mild solution $V = V(t,\xi)$ such that

$$\underline{U}(t,\xi) \le V(t,\xi) \le \overline{U}(t,\xi), \quad \forall\, t \in [0,T],\ \ \xi \in \mathbb{R}.$$

Now, we define an abstract operator $\mathcal{K}$ and a set Γ as follows:

$$\mathcal{K} : V(0,\cdot) \mapsto V(T,\cdot),$$

$$\Gamma := \{\varphi \in \mathcal{BUC}(\mathbb{R},\mathbb{R}) : \underline{U}(0,\xi) \le \varphi(\xi) \le \overline{U}(0,\xi), \xi \in \mathbb{R}\}.$$

It is clear that the operator $\mathcal{K}$ is well defined and

$$\underline{U}(T,\xi) \le V(T,\xi) \le \overline{U}(T,\xi), \quad \xi \in \mathbb{R}.$$

Our purpose is to find a positive T-periodic solution of (2.5). This is equivalent to finding a fixed point of $\mathcal{K}$. Let us postpone this task for the moment and first prove that there is at most one fixed point of $\mathcal{K}$ in a few steps. Assume, for contradiction, that there exist two distinct fixed points V_1 and V_2 of $\mathcal{K}$. We denote by $V_1^*(t,\xi)$, $V_2^*(t,\xi)$ two solutions of (2.5) with the initial value V_1, V_2, respectively. Define $\widehat{V^*}(0,\xi) := V_1^*(0,\xi) - V_2^*(0,\xi) = V_1(\xi) - V_2(\xi)$ and $\widehat{V^*}(t,\xi) := V_1^*(t,\xi) - V_2^*(t,\xi)$. Then, one has

$$\widehat{V^*}(t,\xi) = [\mathcal{S}(t)\widehat{V^*}(0,\cdot)](\xi), \quad \xi \in \mathbb{R},\ \ t > 0.$$

It is easy to check that $\widehat{V^*}(nT,\xi) \ne 0$ for all $n \in \mathbb{N}$, $\xi \in \mathbb{R}$. This contradicts (2.1).

Now, let us prove the existence of the fixed point of $\mathcal{K}$. It is easy to check that $\Gamma \subset \mathcal{B}_\mu(\mathbb{R})$ is nonempty and convex. Thanks to the comparison principle (Lemma

3), T-periodicity of the super-solution $\overline{U}$ and the sub-solution $\underline{U}$, one can check that $\mathcal{K}$ is a mapping from Γ into itself and is continuous in $\mathcal{B}_\mu(\mathbb{R})$.

Furthermore, $\mathcal{K}$ is a compact mapping. Indeed, by [19, Theorem 5.1.2], $V(T,\cdot)$ is in $\mathcal{BC}^{2\theta}(\mathbb{R},\mathbb{R})$ and there exists a constant $C = C(\theta) > 0$, independent of V and ϕ, such that

$$|V(T,\cdot)|_{\mathcal{BC}^{2\theta}(\mathbb{R},\mathbb{R})} \leq C(|V(0,\cdot)|_\infty + D),$$

where $D = \sup_{t\in[0,T],\xi\in\mathbb{R}} |f(t,\xi,\phi,\phi)|$. Then, thanks to the Arzelá–Ascoli theorem, one has the compactness of $\mathcal{K} : \Gamma \to \Gamma$. Clearly, Γ is bounded and closed in $\mathcal{B}_\mu(\mathbb{R})$. Hence, by applying Schauder's fixed point theorem, there exists $V^* \in \mathcal{BC}^1([0,T]\times\mathbb{R},\mathbb{R})$ such that

$$V^*(0,\xi) = V^*(T,\xi), \quad \forall\,\xi \in \mathbb{R}, \tag{2.6}$$

where $V^*(t,\xi)$ is the solution of (2.5) with the initial condition $V^*(0,\cdot)$. Moreover, (2.6) implies that (2.5) with initial value $V(0,\xi) = V^*(0,\xi)$ is well defined and $V^*(0,\xi) = V^*(nT,\xi), \forall\,\xi \in \mathbb{R}, n \in \mathbb{N}$. Using similar arguments in [26, Theorem 2.10] and (2.6), we can extend the function V^* so that

$$V^*(t,\xi) = V^*(t+T,\xi), \quad \forall\,t \in \mathbb{R},\ \xi \in \mathbb{R}.$$

Next, let us define an abstract mapping $\mathcal{T}$ and a set Ψ as follows:

$$\mathcal{T} : \phi \mapsto V^*; \quad \Psi := \left\{\varphi \in \overline{\mathcal{B}}_\mu(\mathbb{R}) : \begin{matrix}\underline{U}(t,\xi) \leq \varphi(t,\xi) \leq \overline{U}(t,\xi) \\ \varphi(t,\xi) = \varphi(t+T,\xi)\end{matrix}, t \in \mathbb{R}, \xi \in \mathbb{R}\right\},$$

where $\phi(t,\xi) = \phi(t+T,\xi)$, V^* is the fixed point of $\mathcal{K}$ corresponding to the function ϕ.

Our goal is to find a fixed point of $\mathcal{T}$, which also serves as a solution of (1.2). It is easy to see that Ψ is nonempty and convex. Furthermore, it follows from the periodicity that

$$\mathcal{T} : \Psi \to \Psi.$$

Thanks to [19, Theorem 5.1.2], one has $V^*(0,\cdot) = V^*(T,\cdot) \in \mathcal{BC}^{2\theta}(\mathbb{R},\mathbb{R})$. It then follows from [19, Theorem 5.1.2 (ii)] that $V^* \in \mathcal{BC}^{\theta,2\theta}(\mathbb{R}\times\mathbb{R},\mathbb{R})$. By using a similar technique in [26, Lemmas 2.7 and 2.8], the mapping $\mathcal{T}$ is continuous and compact in the sense of the decay norm $\|\cdot\|_\mu$. In addition, Ψ is bounded and closed in $\overline{\mathcal{B}}_\mu(\mathbb{R})$ with decay norm $\|\cdot\|_\mu$. Applying Schauder's fixed point theorem again, there exists $U \in \Psi$ such that

$$\mathcal{T}(U) = U.$$

Since $U \in \Psi$, we see that U is T-periodic. Using the regularity in [19, Theorems 5.1.2 and 5.1.8], one has

$$U \in C^{1+\frac{\theta}{2},2+\theta}(\mathbb{R}\times\mathbb{R},\mathbb{R}),$$

which implies that U is a classical solution of (1.2). Furthermore, it follows from $U \geq \underline{U} \geq 0$ and the strong maximum principle that $U > 0$. We obtain the desired result. $\square$

Now, we are in a position to prove the main result.

Proof of Theorem 1. Existence of forced wave solution. Let $c > c^*$. Thanks to Theorem 5 and Sec. 2.1, there exists a classical solution $U = U(t, \xi)$ of (1.2) such that

$$\min\{p(t)e^{-k\xi}, u^*(t)\} \geq U(t,\xi) \geq \max\{p(t)e^{-k\xi} - Lp(t)e^{-k_\epsilon \xi}, 0\},$$

which implies the exponential convergence of the forced wave solution as $\xi \to \infty$, uniformly in $t \in [0, T]$.

From now on, we suppose that $\xi \to \rho(t, \xi)$ is non-decreasing. One can check that

$$\begin{cases} U_t \leq dU_{\xi\xi} + cU_\xi + U[\rho(t,\xi) - g(t)U], \\ U(t,\xi) = U(t+T,\xi), \end{cases} \quad t \in \mathbb{R}, \ \ \xi \in \mathbb{R}.$$

By similar arguments to the one in [7, Lemma 3.3] (put $z = -\xi$, $c^0 = -c$ then $c^0 < -c^*$), there exists $W > 0$ such that

$$\begin{cases} W_t = dW_{\xi\xi} + cW_\xi + W[\rho(t,\xi) - g(t)W], \quad t \in \mathbb{R}, \ \ \xi \in \mathbb{R}, \\ W(t,\xi) = W(t+T,\xi), \quad t \in \mathbb{R}, \ \ \xi \in \mathbb{R}, \\ \lim\limits_{\xi \to -\infty} W(t,\xi) = 0 \quad \text{uniformly in } t \in [0,T], \end{cases}$$

and $U \leq W$. Thus,

$$\lim_{\xi \to -\infty} U(t,\xi) = 0 \quad \text{uniformly in } t \in [0,T].$$

This concludes the investigation of the forced wave solution.

Next, we examine the spreading properties of (1.1). Let u be the solution of (1.1) with the initial condition u_0. Suppose $u(0,\cdot) \in C_c(\mathbb{R})$ and $u(0,\cdot) > 0$.

(i) Outer spreading. Consider the following auxiliary problem:

$$\begin{cases} v_t(t,x) = dv_{xx}(t,x) + v(t,x)[\alpha(t) - g(t)v(t,x)], \quad t > 0, \ \ x \in \mathbb{R}, \\ v(0,x) = u_0(0,x), \quad x \in \mathbb{R}. \end{cases} \tag{2.7}$$

By [17, Lemma 2.2] and $\overline{\alpha} > 0$, we get

$$\lim_{t \to \infty} \sup_{|x| > (c^* + \varepsilon)t} v(t,x) = 0 \quad \text{for any } \varepsilon > 0.$$

In addition, it is easy to check that u is a sub-solution of (2.7) for $t > 0$. By the parabolic comparison principle, we obtain $v \geq u$ and conclude the first limit.

(ii) Inner spreading for $0 < c < c^*$. Consider $\varepsilon \in (0, \frac{c^* - c}{2})$ and let $\varepsilon' > 0$ be small enough such that

$$c^* - \varepsilon < 2\sqrt{d(\overline{\alpha}_+ - \varepsilon')} \quad \text{and} \quad c + 2\varepsilon < 2\sqrt{d(\overline{\alpha}_+ - \varepsilon')}. \tag{2.8}$$

This holds since $c + 2\varepsilon < c^* = 2\sqrt{2d\overline{\alpha}_+}$. Using similar arguments in [17, Lemma 2.5], there exists $M > 0$ and $T_1 > 0$ such that

$$u_t(t,x) \geq du_{xx}(t,x) + u(t,x) \\ \times [\rho(t, x - ct) - \varepsilon' - (g(t) + M)u(t,x)], \quad \forall\, t > T_1,\ \ x \in \mathbb{R}.$$

Consider the following auxiliary problem:

$$\begin{cases} w_t(t,x) = dw_{xx}(t,x) + w(t,x)[\rho(t, x - ct) - \varepsilon' - (g(t) + M)w(t,x)], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ \ x \in \mathbb{R}, \\ w(0,x) = u(T_1 + 1, x), \quad x \in \mathbb{R}. \end{cases} \tag{2.9}$$

Put $a(t,\xi) = \rho(t, -\xi)$. One can check that $\xi \mapsto a(t,\xi)$ is non-increasing. Furthermore, we have

$$\alpha_+(t) = \lim_{\xi \to -\infty} a(t,\xi) =: a_-(t); \quad \alpha_-(t) = \lim_{\xi \to \infty} a(t,\xi) =: a_+(t)$$

and

$$\infty > \overline{a}_- := \frac{1}{T}\int_0^T a_-(t)dt > 0 > \frac{1}{T}\int_0^T a_+(t)dt =: \overline{a}_+ > -\infty.$$

In addition, one has

$$\begin{cases} w_t(t,x) = dw_{xx}(t,x) + w(t,x)[a(t, -x - (-c)t) - \varepsilon' - (g(t) + M)w(t,x)], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ \ x \in \mathbb{R}, \\ w(0,x) = u(T_1 + 1, x), \quad x \in \mathbb{R}. \end{cases}$$

Put $y = -x$, $c^0 = -c$, $v(t,y) = w(t,x) = w(t,-y)$. Direct calculations yield

$$\begin{cases} v_t(t,y) = dv_{yy}(t,y) + v(t,y)[a(t, y - c^0 t) - \varepsilon' - (g(t) + M)v(t,y)], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ \ y \in \mathbb{R}, \\ v(0,x) = u(T_1 + 1, x), \quad x \in \mathbb{R}. \end{cases} \tag{2.10}$$

Thanks to the work of [7], the spreading speed c_F^* of (2.10) is defined by

$$c_F^* := 2\sqrt{d(\overline{a}_- - \varepsilon')} > 0.$$

Thus, $c^0 = -c > -c_F^*$. Now, let us recall the result in [7, Theorem 1.3 (iii)] (put $\mu = c_F^* - \varepsilon \in (0, c_F^*)$).

$$\lim_{t \to \infty} \sup_{y \geq (-c_F^* + \varepsilon)t} |v(t,y) - U_{\varepsilon'}^*(t, y - c^0 t)| = 0,$$

where $U_{\varepsilon'}^*$ is the unique T-periodic non-increasing forced wave of (2.10) and $\lim_{\xi \to -\infty} U_{\varepsilon'}^*(t,\xi) = w_{\varepsilon'}^*(t) > 0$ uniformly in $t \in [0,T]$ for which $w_{\varepsilon'}^*$ is the periodic positive solution of $\beta_t = \beta[a_-(t) - \varepsilon' - (g(t)+M)\beta]$. Since $t[-c_F^* + \varepsilon, c^0 - \varepsilon] \subset t[-c_F^* + \varepsilon, \infty)$, we obtain

$$\lim_{t\to\infty} \sup_{(-c_F^*+\varepsilon)t \le y \le (c^0-\varepsilon)t} |U_{\varepsilon'}^*(t, y - c^0 t) - v(t,y)| = 0.$$

Using the monotonicity of $U_{\varepsilon'}^*$ in ξ, we get

$$\begin{aligned}
&\sup_{(-c_F^*+\varepsilon)t \le y \le (c^0-\varepsilon)t} |U_{\varepsilon'}^*(t, y - c^0 t) - v(t,y)| \\
&\quad \ge \sup_{(-c_F^*+\varepsilon)t \le y \le (c^0-\varepsilon)t} [U_{\varepsilon'}^*(t, -\varepsilon t) - v(t,y)] \\
&\quad = U^*(t, -\varepsilon t) - \inf_{(-c_F^*+\varepsilon)t \le y \le (c^0-\varepsilon)t} v(t,y).
\end{aligned}$$

Thus,

$$\liminf_{t\to\infty} \inf_{(-c_F^*+\varepsilon)t \le y \le (c^0-\varepsilon)t} v(t,y) \ge \liminf_{t\to\infty} U_{\varepsilon'}^*(t, -\varepsilon t).$$

Next, without loss of generality, we assume that $\varepsilon' \le \varepsilon''$ for any $\varepsilon' > 0$ satisfying (2.8) for some $\varepsilon'' > 0$. By the standard parabolic comparison principle, we deduce that $w_{\varepsilon'}^*(t) \ge w_{\varepsilon''}^*(t)$ for all $t \in [0,T]$. Thus, there exists $\delta > 0$, independent of ε', such that $w_{\varepsilon'}^*(t) > \delta$ for all $t \in [0,T]$. This leads to the following estimate:

$$\begin{aligned}
\limsup_{t\to\infty}(\delta - U_{\varepsilon'}^*(t, -\varepsilon t)) &< \limsup_{t\to\infty}(w_{\varepsilon'}^*(t) - U_{\varepsilon'}^*(t, -\varepsilon t)) \\
&\le \limsup_{t\to\infty} \sup_{t' \in [0,T]} |w_{\varepsilon'}^*(t') - U_{\varepsilon'}^*(t', -\varepsilon t)| = 0.
\end{aligned}$$

Here, the last inequality comes from the fact that

$$w_{\varepsilon'}^*(t) - U_{\varepsilon'}^*(t,\xi) \le \sup_{t' \in [0,T]} |w_{\varepsilon'}^*(t') - U_{\varepsilon'}^*(t',\xi)|, \quad \forall\, t \in [0,T],\ \ \xi \in \mathbb{R},$$

with the choice $\xi = -\varepsilon t$. Consequently, one has $\liminf_{t\to\infty} U_{\varepsilon'}^*(t, -\varepsilon t) > \delta$, which implies

$$\delta < \liminf_{t\to\infty} \inf_{(-c_F^*+\varepsilon)t \le y \le (c^0-\varepsilon)t} v(t,y) = \liminf_{t\to\infty} \inf_{(c+\varepsilon)t \le x \le (c_F^*-\varepsilon)t} w(t,x).$$

Then, it is known that $t \mapsto u(t+T_1+1, x)$ is a super-solution of (2.9) for any $t > 0$, $x \in \mathbb{R}$. The parabolic comparison principle implies that $u(t+T_1+1, x) \ge w(t,x)$, $t > 0$, $x \in \mathbb{R}$. In the spirit of [17, Lemma 2.5], one has

$$\liminf_{t\to\infty} \inf_{(c+\varepsilon)t \le x \le (c^*-\varepsilon)t} u(t,x) > 0,$$

which concludes the proof. $\square$

Acknowledgment

Part of this work had been done while the authors were visiting the Vietnam Institute for Advanced Study in Mathematics, whose hospitality and financial support have been acknowledged.

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