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Results in Mathematics

Second-Order Necessary and Sufficient Optimality Conditions under Asymptotic Cones for Optimization Problems and Applications to Control Optimal Problems

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Abstract. This paper investigates the optimality conditions for mathematical programming problems involving geometric and functional constraints, with objective functions that are Fréchet differentiable and their gradient mappings are locally Lipschitz on an open set. We first establish formulas to compute the asymptotic second-order tangent cone of the constraint sets and decompose the asymptotic second-order tangent cone for the intersection of these sets. We then derive second-order necessary and sufficient optimality conditions for mathematical programming problems. The results are applied to a class of optimal control problems.

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1. Introduction

The theory of optimality conditions, a key area in variational analysis and optimization, captivates many researchers; cf., e.g., [4–6, 9–13, 16, 28, 30, 31, 36, 37, 39]. Hiriart-Urruty et al. [9] and Toan et al. [37] utilized the generalized Hessian matrix to establish second-order necessary conditions for a class of differentiable mathematical programming problems. Using a *stricter regularity* condition than the corresponding regularity condition in [37], Toan et al. [36] established second-order necessary and sufficient optimality conditions for the

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mathematical programming problem in [37]. When the objective function may not be differentiable, Toan et al. [39] derived directional *first-order necessary* optimality conditions for the mathematical programming problem.

Gutiérrez et al. [11,12] employed the Hadamard parabolic and asymptotic derivative to derive second-order necessary and sufficient optimality conditions for general mathematical programming problems. Huy et al. demonstrated in Section 4 of [10] that asymptotic and the second-order parabolic derivatives may not exist. The authors in [10] introduced the second-order symmetric subdifferential and its associated calculus rules to derive second-order necessary and sufficient optimality conditions for differentiable mathematical programming problems with *geometric* constraints. As noted in [28], the second-order symmetric subdifferential can be strictly smaller than the generalized Hessian matrix. Nevertheless, second-order necessary and sufficient optimality conditions for mathematical programming problems with both *geometric and functional* constraints described by target sets remain unexplored in this context.

In this work, by establishing formulas to compute the asymptotic second-order tangent cone of the constraint sets and decompose the asymptotic second-order tangent cone for the intersection of these sets, we derive second-order necessary and sufficient optimality conditions for mathematical programming problems with both *geometric and functional* constraints described by target sets. We can say that this study extends Huy and Tuyen s results in [10]. Moreover, we *only use the regularity* condition as in [37,39] to achieve sufficient optimality conditions for mathematical programming problems, even if the *objective function may not be second-order differentiable*. Thus, our results also serve to develop and refine those of [36,39].

The paper is organized as follows. Section 2 establishes auxiliary results concerning formulas to compute the asymptotic second-order tangent cone of the constraint sets and decompose the asymptotic second-order tangent cone for the intersection of these sets, essential for the using in later sections. In this section, we also derive the rules for decomposition of tangent sets (cones) for the Cartesian product of sets and the symmetric second-order subdifferential for the sum of functions in $C^{1,1}$. The main results are Theorem 3.1 and Theorem 3.4 on second-order necessary and sufficient optimality conditions for a mathematical programming problem, presented in Section 3. At the final section, we give an example to illustrate the main results. In this section, we also discuss for the application of our main results to a discrete optimal control problem with mixed constraints in the case where the objective function may not be twice differentiable.

2. Preliminaries and Auxiliary Results

2.1. Preliminaries

In this section, we review key concepts and facts from variational analysis and generalized differentiation that will be utilized later. These notations and facts are detailed in [6,8,18,23,28,29,34].

Let E_1 and E_2 be finite-dimensional Euclidean spaces and $F : E_1 \rightrightarrows E_2$ be a multifunction. The effective *domain*, denoted by dom F and the *graph* of F, denoted by gph F, are defined as

$$\text{dom } F := \{z \in E_1 : F(z) \neq \emptyset\}$$

and

$$\text{gph } F := \{(z, v) \in E_1 \times E_2 : v \in F(z)\}.$$

The set

$$\text{Lim sup}_{z \to \bar{z}} F(z) = \{z^* \in E_1 : \exists z_n \to \bar{z}, z_n^* \to z^* \text{ with } z_n^* \in F(z_n) \ \ \forall n \in \mathbb{N}\}$$

is called the *Kuratowski-Painlevé upper limit* of F at $\bar{z}$.

Let E be a finite-dimensional Euclidean space, D be a nonempty closed subset of E, and $\bar{z} \in D$. The *closure, interior, conic hull*, and *convex hull* of D are denoted, respectively, by cl D, int D, cone D, and conv D. The set

$$T(D; \bar{z}) = \text{Lim sup}_{t \to 0^+} \frac{D - \bar{z}}{t} = \big\{h \in E : \exists t_n \to 0^+, \exists h_n \to h \text{ such that } \bar{z} + t_n h_n \in D \ \ \forall n \in \mathbb{N}\}$$

is called the *tangent cone* to D at $\bar{z}$. The *adjacent tangent cone* to D at $\bar{z}$ is defined by

$$T^b(D; \bar{z}) = \text{Lim inf}_{t \to 0^+} \frac{D - \bar{z}}{t} = \big\{h \in E : \forall t_n \to 0^+, \exists h_n \to h \text{ such that } \bar{z} + t_n h_n \in D \ \ \forall n \in \mathbb{N}\big\}.$$

The set

$$C(D; \bar{z}) = \text{Lim inf}_{t \to 0^+, \ z \xrightarrow{D} \bar{z}} \frac{D - \bar{z}}{t} = \Big\{h \in E : \forall z_n \xrightarrow{D} \bar{z}, \forall t_n \to 0^+, \exists h_n \to h \text{ such that } z_n + t_n h_n \in D \ \ \forall n \in \mathbb{N}\}$$

is called the *Clarke tangent cone* to D at $\bar{z}$.

It is well known that if D is a convex set, then

$$T(D; \bar{z}) = T^b(D; \bar{z}) = C(D; \bar{z}) = \text{cl}\big(D(\bar{z})\big),$$

where

$$D(\bar{z}) := \text{cone}(D - \bar{z}) = \{\lambda(d - \bar{z}) : d \in D, \lambda > 0\}.$$

The *second-order tangent set* to D at $\bar{z}$ in the direction $v \in E$ is defined by

$$T^2(D;\bar{z},v) = \text{Lim sup}_{t\to 0^+} \frac{D-\bar{z}-tv}{\frac{t^2}{2}}$$
$$= \left\{ w : \exists t_n \to 0^+, \exists w_n \to w \text{ such that } \bar{z}+t_nv+\frac{t_n^2}{2}w_n \in D \ \ \forall n \in \mathbb{N} \right\}.$$

The set

$$T^{2b}(D;\bar{z},v) = \text{Lim inf}_{t\to 0^+} \frac{D-\bar{z}-tv}{\frac{t^2}{2}}$$
$$= \left\{ w : \forall t_n \to 0^+, \exists w_n \to w \text{ such that } \bar{z}+t_nv+\frac{t_n^2}{2}w_n \in D \ \ \forall n \in \mathbb{N} \right\}$$

is called the *adjacent second-order tangent set* to D at $\bar{z}$ in the direction $v \in E$.

When D is a convex set and $v \in D(\bar{z}) = \text{cone}(D-\bar{z})$, then

$$T^2(D;\bar{z},v) = T^{2b}(D;\bar{z},v) = T\big(T(D;\bar{z});v\big)$$

by [18, Lemma 2.3]. Moreover, if D is a polyhedral convex set, then we also have $T^2(D;\bar{z},v) = T\big(T(D;\bar{z});v\big)$ for all $v \in T(D;\bar{z})$.

The *asymptotic second-order tangent cone* to D at $\bar{z}$ in the direction $v \in E$ is

$$T^{''}(D;\bar{z},v) = \Big\{ w : \exists (t_n,r_n) \to (0^+,0^+), \exists w_n \to w \text{ such that } \frac{t_n}{r_n} \to 0 \text{ and}$$
$$\bar{z}+t_nv+\frac{1}{2}t_nr_nw_n \in D \ \ \forall n \in \mathbb{N} \Big\}.$$

The cone

$$T^{''b}(D;\bar{z},v) = \Big\{ w : \forall (t_n,r_n) \to (0^+,0^+), \exists w_n \to w \text{ such that } \frac{t_n}{r_n} \to 0 \text{ and}$$
$$\bar{z}+t_nv+\frac{1}{2}t_nr_nw_n \in D \ \ \forall n \in \mathbb{N} \Big\}$$

is called the *asymptotic adjacent second-order cone* to D at $\bar{z}$ in the direction $v \in E$.

When D is a convex set and $v \in D(\bar{z}) = \text{cone}(D-\bar{z})$, then there exists $\lambda > 0$ such that $v = \lambda(z-\bar{z})$ for some $z \in D$. By the convexity of D, for any $(t_n,r_n) \to (0^+,0^+)$, $\frac{t_n}{r_n} \to 0$, we have

$$t_nv = t_n\lambda z + (1-t_n\lambda)\bar{z} - \bar{z} \in D - \bar{z}.$$

This implies that $\bar{z}+t_nv \in D$, and so, $0 \in T^{''b}(D;\bar{z},v)$.

The cone T and the sets T^2, T^{2b} are well known. The cone $T^{''}$ was first introduced by Penot in [32]. We refer the readers to [17, Section 4.11.3] for the definition and discussions related to cone $T^{''b}$. If D is a convex set and $v \in T(D;\bar{z})$ such that $T^{''}(D;\bar{z},v) \neq \emptyset$, one has

$$T^2(D;\bar{z},v) \subset T^{''}(D;\bar{z},v) = \text{cl cone}[\text{cone}(D-\bar{z}) - v].$$

In general cases, there is no inclusion relation between $T^2(D;\bar{z},v)$ and $T^{''}(D;\bar{z},v)$.

The set

$$\widehat{N}_\varepsilon(D;\bar{z}) := \left\{ z^* \in E : \limsup_{z \xrightarrow{D} \bar{z}} \frac{\langle z^*, z-\bar{z}\rangle}{\|z-\bar{z}\|} \leq \varepsilon \right\}$$

is called the $\varepsilon-$ *Fréchet normal set* to D at $\bar{z}$. When $\varepsilon = 0$, the set $\widehat{N}(D;\bar{z}) := \widehat{N}_0(D;\bar{z})$ is called the *Fréchet normal cone* to D at $\bar{z}$. If $\bar{z} \notin D$ one puts $\widehat{N}_\varepsilon(D;\bar{z}) := \emptyset$. The set

$$N_C(D;\bar{z}) := \big(C(D;\bar{z})\big)^* = \{z^* \in E : \langle z^*, z\rangle \leq 0 \text{ whenever } z \in C(D;\bar{z})\}$$

is called the *Mordukhovich normal cone* to D at $\bar{z}$. The set D is said to be *normally regular* at $\bar{z} \in D$ if $\widehat{N}(D;\bar{z}) = N(D;\bar{z})$. The set

$$N_C(D;\bar{z}) := \big(C(D;\bar{z})\big)^* = \{z^* \in E : \langle z^*, z\rangle \leq 0 \text{ whenever } z \in C(D;\bar{z})\}$$

is called the *Clarke normal cone* to D at $\bar{z}$.

It is also known that if D is a convex set, then the Fréchet normal cone coincides with Mordukhovich normal cone, coincides with Clarke normal cone, and coincides with normal cone of convex analysis for convex sets.

Note that the Clarke normal cone to D at $\bar{z}$ always coincides with the convex closure of the Mordukhovich normal cone, i.e., $N_C(D;\bar{z}) = \text{cl conv } N(D;\bar{z})$.

We denote the set of *perpendicular vectors* to $z \in E$ in E^* by

$$z^\perp = \{z^* \in E^* : \langle z^*, z\rangle = 0\}.$$

Let $F : E_1 \rightrightarrows E_2$ be a multifunction, $(\bar{z},\bar{y}) \in \text{cl gph } F$. The multifunction $D^*F(\bar{z},\bar{y}) : E_2 \rightrightarrows E_1$, defined by

$$D^*F(\bar{z},\bar{y})(y^*) := \{z^* \in E_1 : (z^*, -y^*) \in N\big((\bar{z},\bar{y}); \text{gph } F\big)\}, \quad \forall y^* \in E_2$$

is called the *Mordukhovich coderivative* of F at the point $(\bar{z},\bar{y})$. The symbol $D^*F(\bar{z})$ is used when F is single-valued at $\bar{z}$ and $\bar{y} = F(\bar{z})$.

Let $\varphi : E \to \bar{R}$ be an extended real-valued function and $\bar{z} \in E$ be such that $\varphi(\bar{z})$ is finite. For each $\varepsilon \geq 0$, the set

$$\hat{\partial}_\varepsilon \varphi(\bar{z}) := \left\{ z^* \in E : \liminf_{z \to \bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z-\bar{z}\rangle}{\|z-\bar{z}\|} \geq -\varepsilon \right\}$$

is called the $\varepsilon-$ *Fréchet subdifferential* of φ at $\bar{z}$. The set $\widehat{\partial}\varphi(\bar{z}) = \widehat{\partial}_0\varphi(\bar{z})$ is called the Fréchet subdifferential of φ at $\bar{z}$ and the set

$$\partial\varphi(\bar{z}) := \operatorname*{Lim\,sup}_{\substack{z \xrightarrow{\Omega} \bar{z} \\ \varepsilon\downarrow 0}} \widehat{\partial}_\varepsilon\varphi(z)$$

is called the *Mordukhovich subdifferential* of φ at $\bar{z}$. It is known that the Mordukhovich subdifferential reduces to the classical Fréchet derivative for strictly differentiable functions and to subdifferential of convex analysis for convex functions. The set

$$\partial^+\varphi(\bar{z}) := -\partial(-\varphi)(\bar{z})$$

and

$$\partial_S(\varphi)(\bar{z}) := \partial(\varphi)(\bar{z}) \cup \partial^+(\varphi)(\bar{z})$$

is called the *upper subdifferential* and the *symmetric subdifferential* of φ at $\bar{z}$, respectively.

The notation of the symmetric subdifferential was first introduced by Kruger and Mordukhovich [28]. It is easy to see that $\partial_S(\lambda\varphi)(\bar{z}) = \lambda\partial_S(\varphi)(\bar{z})$ for all $\lambda \in \mathbb{R}$. We also have the following result from the definition of the symmetric subdifferential.

Proposition 2.1. *Suppose that $f : E \to \mathbb{R}$ is strictly differentiable at $\bar{z}$ and $g : E \to \mathbb{R}$ is Lipschitz continuous around $\bar{z}$. Then,*

$$\partial_S(f + g)(\bar{z}) = \nabla f(\bar{z}) + \partial_S g(\bar{z}).$$

Let $(\bar{z}, \bar{y}) \in \text{gph}\ \partial\varphi$. The *Mordukhovich second-order subdifferential* of φ at $\bar{z}$ relative to $\bar{y}$ is a multifunction $\partial^2\varphi(\bar{z}, \bar{y}) : E \rightrightarrows E$ defined by

$$\partial^2\varphi(\bar{z}, \bar{y})(u) := (D^*\partial\varphi)(\bar{z}, \bar{y})(u) = \Big\{v : (v, -u) \in N\big(\text{gph}\ \partial\varphi; (\bar{z}, \bar{y})\big)\Big\}, \quad \forall u \in E.$$

Let D be an open subset of E. We denote by $C^{1,1}(D)$ the class of all real-valued functions g, which are Fréchet differentiable on D, and whose gradient mapping $\nabla g(\cdot)$ is locally Lipschitz on D. When $g \in C^{1,1}(D)$ and $\bar{z} \in D$, we get from [28, Theorem 1.90] that

$$\partial^2 g(\bar{z})(u) = \partial^2 g\big(\bar{z}, \nabla g(\bar{z})\big)(u) = \partial\langle u, \nabla g\rangle(\bar{z}), \quad \forall u \in E.$$

Let $g \in C^{1,1}(D)$ and $\bar{z} \in D$. The *symmetric second-order subdifferential* of g at $\bar{z}$ is a multifunction $\partial^2_S : E \rightrightarrows E$ defined by

$$\partial^2_S g(\bar{z})(u) = \partial^2_S g\big(\bar{z}, \nabla g(\bar{z})\big)(u) = \partial\langle u, \nabla g\rangle(\bar{z}) \cup \partial^+\langle u, \nabla g\rangle(\bar{z}), \quad \forall u \in E.$$

The definition of the second-order subdifferential of a function via a coderivative of the first-order subdifferential was suggested in [27]. Now, we have the following properties of the symmetric second-order subdifferential directly from the definition.

Proposition 2.2. *Let $g \in C^{1,1}(D)$ and $\bar{z} \in D$. The following assertions hold:*
(i) *For any $\lambda \in \mathbb{R}$ one has $\partial_S^2 g(\bar{z})(\lambda z) = \lambda \partial_S^2 g(\bar{z})(z) \quad \forall z \in E$;*
(ii) *For any $\lambda \in \mathbb{R}$ one has $\partial_S^2 (\lambda g)(\bar{z})(z) = \lambda \partial_S^2 g(\bar{z})(z) \quad \forall z \in E$;*
(iii) *For any $z \in E$ the mapping $z \mapsto \partial_S^2 g(\bar{z})(z)$ is locally bounded. Moreover, if $z_n \to \bar{z}$, $z_n^* \to z^*$ and $z_n^* \in \partial_S^2 g(z_n)(z)$ for all $n \in \mathbb{N}$, then $z^* \in \partial_S^2 g(\bar{z})(z)$.*

2.2. Auxiliary Results

In this subsection, we suppose that E_1, E_2, E_3 are finite-dimensional Euclidean spaces. Assume moreover that two functions $G_1 : E_1 \to E_2, G_2 : E_2 \to E_3$ have second-order Gâteaux derivatives around $z_0 \in E_1$, and $C \subset E_2$ is a nonempty closed convex set. Put

$$\Omega = \{z \in Z : G_2(z) = 0\}.$$

Given $z_0 \in \Omega \cap G_1^{-1}(C)$ and $v \in E_1$, this subsection gives formulas to compute the asymptotic second-order tangent cones $T^{''}(\Omega; z_0, v), T^{''}(\Omega \cap G_1^{-1}(C); z_0, v)$ and the asymptotic adjacent second-order cone $T^{''b}(\Omega; z_0, v), T^{''b}(\Omega \cap G_1^{-1}(C); z_0, v)$. We start with the following lemma.

Lemma 2.3. *Assume that $z_0 \in \Omega \cap G_1^{-1}(C)$, $\nabla G_2(\cdot)$ is continuous at z_0. Suppose further that $\nabla G_2(z_0)$ is surjective and the following regularity condition is satisfied:*

$$\bigcap_{z \in B(z_0, r) \cap \Omega} \Big[\nabla G_1(z_0)\big(T(\Omega; z)\big) - C\big(G_1(z_0)\big)\Big] = E_2 \quad \text{for some } r > 0. \tag{1}$$

Then, for each $v \in E_1$

$$\begin{aligned} T^{''b}(\Omega \cap G_1^{-1}(C); z_0, v) = & T^{''b}(\Omega; z_0, v) \cap [\nabla G_1(z_0)^{-1} \\ & \big(T^{''b}(C; G_1(z_0), \nabla G_1(z_0)v)\big)]. \end{aligned}$$

Proof. We will use some arguments as in the proof of [6, Theorem 3.1] and [18, Theorem 2.3]. Take any $w \in T^{''b}(\Omega \cap G_1^{-1}(C); z_0, v)$. By the definition of the asymptotic adjacent second-order tangent cone, for all $(t_k, r_k) \to (0^+, 0^+)$, there exists $w_k \to w$ such that $\frac{t_k}{r_k} \to 0$ and $z_0 + t_k v + \frac{1}{2} t_k r_k w_k \in \Omega \cap G_1^{-1}(C), \forall k \in \mathbb{N}$. It follows that

$$G_1(z_0 + t_k v + \frac{1}{2} t_k r_k w_k) \in C, \ \forall k \in \mathbb{N}.$$

By Taylor expansion, we get

$$G_1(z_0+t_kv+\frac{1}{2}t_kr_kw_k)=G_1(z_0)+t_k\nabla G_1(z_0)v + \frac{1}{2}t_kr_k\Big[\nabla G_1(z_0)w_k+2\frac{t_k}{r_k}\nabla^2G_1(z_0)vv) + \frac{o(t_kr_k)}{t_kr_k}\Big]$$
$$= G_1(z_0)+t_k\nabla G_1(z_0)v + \frac{1}{2}t_kr_k\Big[\nabla G_1(z_0)w_k+\frac{o(t_kr_k)}{t_kr_k}\Big]\in C,$$

for all $k\in\mathbb{N}$. This implies that

$$\nabla G_1(z_0)w\in T^{''b}(C;G_1(z_0),\nabla G_1(z_0)v).$$

It is equivalent to

$$w\in\nabla G_1(z_0)^{-1}\big(T^{''b}(C;G_1(z_0),\nabla G_1(z_0)v)\big).$$

Moreover, it is easy to see that

$$T^{''b}(\Omega\cap G_1^{-1}(C);z_0,v)\subset T^{''b}(\Omega;z_0,v),$$

for all $v\in Z$. Thus, we obtain

$$T^{''b}(\Omega\cap G_1^{-1}(C);z_0,v)\subset T^{''b}(\Omega;z_0,v)\cap[\nabla G_1(z_0)^{-1} \big(T^{''b}(C;G_1(z_0),\nabla G_1(z_0)v)\big)].$$

Conversely, take any

$$w\in T^{''b}(\Omega;z_0,v)\cap[\nabla G_1(z_0)^{-1}\big(T^{''b}(C;G_1(z_0),\nabla G_1(z_0)v)\big)].$$

So, for all $(t_n,r_n)\to(0^+,0^+)$, there exists $w_n\to w$ such that $\frac{t_n}{r_n}\to 0$,

$$z_0+t_nv+\frac{1}{2}t_nr_nw_n\in\Omega,\ \forall n\in\mathbb{N}$$

and

$$d\Big(G_1(z_0)+t_n\nabla G_1(z_0)v+\frac{t_nr_n}{2}\nabla G_1(z_0)w,C\Big)=o(t_nr_n),\forall n\in\mathbb{N}.$$

By [18, Theorem 2.2], there are positive numbers k_1 and ε such that

$$d(z,\Omega\cap G_1^{-1}(C))\le k_1d\big(G_1(z),C\big),$$

for all $z\in B_Z(z_0,\varepsilon)\cap\Omega$. This follows that

$$d\big(z_0+t_nv+\frac{t_nr_n}{2}w_n,\Omega\cap G_1^{-1}(C)\big)\le k_1d\big(G_1(z_0+t_nv+\frac{t_nr_n}{2}w_n),C\big),$$

for all n large enough. By Taylor expansion, we get

$$\begin{aligned}G_1(z_0+t_nv+\frac{t_nr_n}{2}w_n)&=G_1(z_0)+t_n\nabla G_1(z_0)v+\frac{t_nr_n}{2}\nabla G_1(z_0)w_n+o(\frac{t_nr_n}{2})\\&=G_1(z_0)+t_n\nabla G_1(z_0)v+\frac{t_nr_n}{2}\nabla G_1(z_0)w\\&\qquad+\frac{t_nr_n}{2}\big(\nabla G_1(z_0)w_n-\nabla G_1(z_0)w\big)+o(\frac{t_nr_n}{2}),\end{aligned}$$

for all $n\in\mathbb{N}$. So,

$$\begin{aligned}d\Big(G_1(z_0+t_nv+\frac{t_nr_n}{2}w_n),C\Big)\le\,&d\Big(G_1(z_0)+t_n\nabla G_1(z_0)v+\frac{t_nr_n}{2}\nabla G_1(z_0)w,C\Big)\\&+\|\frac{t_nr_n}{2}\big(\nabla G_1(z_0)w_n-\nabla G_1(z_0)w\big)+o(\frac{t_nr_n}{2})\|\\\le\,&o(t_nr_n)+\frac{t_nr_n}{2}\|\nabla G_1(z_0)w_n-\nabla G_1(z_0)w\|+o(t_nr_n)=o(t_nr_n),\end{aligned}$$

for all n large enough. Hence,

$$d\big(z_0+t_nv+\frac{t_nr_n}{2}w_n,\Omega\cap G_1^{-1}(C)\big)=o(t_nr_n),\text{for all } n \text{ large enough.}$$

Thus, $w\in T^{''b}(\Omega\cap G_1^{-1}(C);z_0,v)$. The proof of the lemma is complete. □

A key improvement over [6, Theorem 3.1] is that the above lemma only requires the set Ω to be closed, whereas [6] required Ω to be both closed and convex. This sharper version plays an important role in establishing the Lagrange multiplier rule. When set Ω is not necessarily convex, Kien and Nhu [18, Theorem 2.3] established formulas to decompose the adjacent second-order tangent set for $\Omega\cap G_1^{-1}(C)$. By applying arguments similar to those in Kien and Nhu's proof, we obtain the above result for decomposition the asymptotic adjacent second-order cone of $\Omega\cap G_1^{-1}(C)$.

To obtain the dual form of the optimality conditions, we must calculate $T^{''}(\Omega;z_0,v), T^{''b}(\Omega;z_0,v)$ and $T^{''}(C;z_0,v), T^{''b}(C;z_0,v)$. The following results provide the corresponding formulas.

Lemma 2.4. *Assume that $z_0\in\Omega$, $\nabla G_2(\cdot)$ is continuous at z_0 and $\nabla G_2(z_0)$ is surjective.*

If $v\in T(\Omega;z_0)$, then

$$T^{''}(\Omega;z_0,v)=T^{''b}(\Omega;z_0,v)=T(\Omega;z_0)=\big\{w\in Z:\nabla G_2(z_0)w=0\big\}.$$

Proof. We will employ techniques from the proof of [18, Lemma 2.2]. Take any $v\in T(\Omega;z_0)$ and $w\in T^{''b}(\Omega;z_0,v)$. Then, for all $(t_k,r_k)\to(0^+,0^+)$, there is $w_k\to w$ such that $\frac{t_k}{r_k}\to 0$ and $z_0+t_kv+\frac{1}{2}t_kr_kw_k\in\Omega$, for all $k\in\mathbb{N}$. So, $G_2(z_0+t_kv+\frac{1}{2}t_kr_kw_k)=G_2(z_0)=0$, for all $k\in\mathbb{N}$. Note that $T(\Omega;z_0)=\big\{w\in Z:\nabla G_2(z_0)w=0\big\}$. Hence,

$$\frac{G_2(z_0+t_kv+\frac{1}{2}t_kr_kw_k)-G_2(z_0+t_kv)}{\frac{t_kr_k}{2}} + \frac{G_2(z_0+t_kv)-G_2(z_0)-t_k\nabla G_2(z_0)v}{\frac{t_kr_k}{2}} = 0,$$

for all $k\in\mathbb{N}$. We have

$$\lim_{k\to\infty}\frac{G_2(z_0+t_kv+\frac{1}{2}t_kr_kw_k)-G_2(z_0+t_kv)}{\frac{t_kr_k}{2}} = \nabla G_2(z_0)w$$

and

$$\begin{aligned}&\lim_{k\to\infty}\frac{G_2(z_0+t_kv)-G_2(z_0)-t_k\nabla G_2(z_0)v}{\frac{t_kr_k}{2}}\\ &=\lim_{k\to\infty}\frac{G_2(z_0)+\nabla G_2(z_0)t_kv+\frac{1}{2}\nabla^2G_2(z_0)(t_kv)^2+o(t_kv)^2-G_2(z_0)-t_k\nabla G_2(z_0)v}{\frac{t_kr_k}{2}}\\ &=\nabla^2G_2(z_0)v^2\lim_{k\to\infty}\frac{t_k}{r_k}+\lim_{k\to\infty}\frac{o(t_kv)^2}{\frac{t_kr_k}{2}}=0.\end{aligned}$$

So,

$$T^{''b}(\Omega;z_0,v)\subset\{w\in Z:\nabla G_2(z_0)w=0\}=T(\Omega;z_0),\quad \text{for all } v\in T(\Omega;z_0).$$

Conversely, let $v\in T(\Omega;z_0)$ and w belong to the right-hand side. Then, for $(t_n,r_n)\to(0^+,0^+)$ such that $\frac{t_n}{r_n}\to 0$, we have

$$\begin{aligned}0=\nabla G_2(z_0)w&=\nabla^2G_2(z_0)vv\lim_{n\to\infty}\frac{t_n}{r_n}\\ &=\lim_{n\to\infty}\frac{G_2(z_0+t_nv)-G_2(z_0)-t_n\nabla G_2(z_0)v}{\frac{t_nr_n}{2}}.\end{aligned}$$

Note that $\nabla G_2(z_0)$ is surjective. By the Ljusternik theorem [14, page 30], there is a neighborhood V of z_0, a number $\ell>0$, and a mapping $\Upsilon: V\to Z$ such that

$$G_2\big(\zeta+\Upsilon(\zeta)\big)=G_2(z_0), \|\Upsilon(\zeta)\|\le\ell\|G_2(\zeta)-G_2(z_0)\|,\ \text{for all } \zeta\in V.$$

Put $\zeta_n=z_0+t_nv+\frac{t_nr_n}{2}w$ for each $n\in\mathbb{N}$. Then,

$$0=G_2\big(z_0+t_nv+\frac{t_nr_n}{2}w+\Upsilon(\zeta_n)\big)=G_2\Big(z_0+t_nv+\frac{t_nr_n}{2}\big(w+\frac{\Upsilon(\zeta_n)}{\frac{t_nr_n}{2}}\big)\Big),$$

for n large enough. Put $w_n = w + \frac{\Upsilon(\zeta_n)}{\frac{t_n r_n}{2}}$. We will show that $w_n \to w$ or $\frac{\Upsilon(\zeta_n)}{\frac{t_n r_n}{2}} \to 0$. Indeed, we get

$$\begin{aligned}\|\frac{\Upsilon(\zeta_n)}{\frac{t_n r_n}{2}}\| &\leq \ell\Big\|\frac{G_2\big(z_0+t_nv+\frac{t_nr_n}{2}w\big)-G_2(z_0)}{\frac{t_nr_n}{2}}\Big\|\\ &= \ell\Big\|\frac{G_2\big(z_0+t_nv+\frac{t_nr_n}{2}w\big)-G_2(z_0+t_nv)}{\frac{t_nr_n}{2}}\\ &\qquad + \frac{G_2\big(z_0+t_nv\big)-G_2(z_0)-t_n\nabla G_2(z_0)v}{\frac{t_nr_n}{2}}\Big\|,\end{aligned}$$

with $v \in T(\Omega; z_0)$. This follows that

$$\begin{aligned}\lim_{n\to\infty}\|\frac{\Upsilon(\zeta_n)}{\frac{t_n r_n}{2}}\| &\leq \ell\Big\|\lim_{n\to\infty}\frac{G_2\big(z_0+t_nv+\frac{t_nr_n}{2}w\big)-G_2(z_0+t_nv)}{\frac{t_nr_n}{2}}\\ &\qquad + \lim_{n\to\infty}\frac{G_2\big(z_0+t_nv\big)-G_2(z_0)-t_n\nabla G_2(z_0)v}{\frac{t_nr_n}{2}}\Big\|\\ &= \ell\|\nabla G_2(z_0)w+\nabla^2G_2(z_0)vv\lim_{n\to\infty}\frac{t_n}{r_n}\| = \ell\|\nabla G_2(z_0)w\| = 0.\end{aligned}$$

Hence, we have showed that $w_n \to w$ and $z_0 + t_nv + \frac{t_nr_n}{2}w_n \in \Omega$. So, $w \in T^{''b}(\Omega; z_0, v)$. Thus,

$$T^{''b}(\Omega; z_0, v) = T(\Omega; z_0) = \big\{w \in Z : \nabla G_2(z_0)w = 0\big\},$$

for all $v \in T(\Omega; z_0)$.

By similar arguments, we can show that

$$T^{''}(\Omega; z_0, v) = T(\Omega; z_0) = \big\{w \in Z : \nabla G_2(z_0)w = 0\big\},$$

for all $v \in T(\Omega; z_0)$. The proof of the lemma is complete. □

Lemma 2.5. *Assume that $z_0 \in C$, $v \in C(z_0)$. Then*

$$T^{''}(C; z_0, v) = T^{''b}(C; z_0, v) = T\big(T(C; z_0); v\big).$$

Proof. We will apply certain arguments similar to those used in the proof of [6, Proposition 3.1]. We first prove that

$$w + T\big(T(C; z_0); v\big) \subset T^{''b}(C; z_0, v) \subset T\big(T(C; z_0); v\big), \tag{2}$$

for all $w \in T^{''b}(C; z_0, v)$. Indeed, for each $w \in T^{''b}(C; z_0, v)$ and choose $w_{tr} \to w$ with $z_0 + tv + \frac{tr}{2}w_{tr} \in C$ for all $t > 0,\ r > 0$ and $t = o(r)$ as $r \to 0^+$. Let $y \in C$ and $\lambda, \mu \in \mathbb{R}^+$. Then, for all r sufficiently small, $t = o(r)$ as $r \to 0^+$, we get

$$\big(1 - \frac{\lambda r}{2}\big)\big[z_0 + tv + \frac{tr}{2}w_{tr}\big] + \frac{\lambda r}{2}\big[z_0 + \mu t(y - z_0)\big] \in C.$$

So,

$$z_0 + tv + \frac{tr}{2}\big[\big(1 - \frac{\lambda r}{2}\big)w_{tr} + \lambda\mu(y - z_0) - \lambda v\big] \in C.$$

This follows that

$$w + \lambda\big(\mu(y - z_0) - v\big) = w + \lambda\mu(y - z_0) - \lambda v \in T^{''b}(C; z_0, v).$$

From C is convex, $T(C; z_0) = \mathrm{cl}\big(\mathrm{cone}(C - z_0)\big)$ and $T^{''b}(C; z_0, v)$ is closed by [17, Theorem 4.11.11], we get

$$w + \lambda\big[T(C; z_0) - v\big] \subset T^{''b}(C; z_0, v).$$

Note that $T(C; z_0)$ is also convex. So,

$$w + T\big(T(C; z_0); v\big) \subset T^{''b}(C; z_0, v).$$

We now prove the second inclusion of (2). Note that $C \subset z_0 + T(C; z_0)$. So,

$$d\big(z_0 + tv + \frac{tr}{2}w; C\big) \geq td\big(v + \frac{r}{2}w; T(C; z_0)\big).$$

Hence, $w \in T^{''b}(C; z_0, v)$ follows that $d\big(v + \frac{r}{2}w; T(C; z_0)\big) \sim o(r)$ as $r \to 0^+$. This implies that $\frac{1}{2}w \in T\big(T(C; z_0); v\big)$. Note that $T\big(T(C; z_0); v\big)$ is a cone. So, $w \in T\big(T(C; z_0); v\big)$. Thus, the inclusion (2) is valid. From $v \in C(z_0)$, we have $0 \in T^{''b}(C; z_0, v)$. Thus,

$$T^{''b}(C; z_0, v) = T\big(T(C; z_0); v\big) \subset T^{''}(C; z_0, v).$$

Conversely, take any $w \in T^{''}(C; z_0, v)$. Then, there are $(t_k, r_k) \to (0^+, 0^+)$ and $w_k \to w$ such that $\frac{t_k}{r_k} \to 0$ and $z_0 + t_k v + \frac{1}{2}t_k r_k w_k = z_0 + t_k(v + \frac{r_k}{2}w_k) \in C$, for all $k \in \mathbb{N}$. This implies that

$$v + \frac{r_k}{2}w_k \in T(C; z_0).$$

It follows that

$$w_k \in T\big(T(C; z_0); v\big).$$

Letting $k \to \infty$, we get $w \in T\big(T(C; z_0); v\big)$. Thus,

$$T^{''}(C; z_0, v) \subset T\big(T(C; z_0); v\big) = T^{''b}(C; z_0, v).$$

The proof of the lemma is complete. □

Since lemmas 2.3 - 2.5, we have the following corollary.

Corollary 2.6. *Assume that $z_0 \in \Omega \cap G_1^{-1}(C)$, $\nabla G_2(\cdot)$ is continuous at z_0. Suppose further that $\nabla G_2(z_0)$ is surjective and the following regularity condition is satisfied:*

$$\bigcap_{z \in B(z_0, r) \cap \Omega} \Big[\nabla G_1(z_0)\big(T(\Omega; z) - C\big(G_1(z_0)\big)\Big] = E_2 \quad \text{for some } r > 0.$$

Then for each $v \in T(\Omega; z_0)$, $\nabla G_1(z_0)v \in C\big(G_1(z_0)\big)$, *one has*

$$\begin{aligned} T^{''}(\Omega \cap G_1^{-1}(C); z_0, v) &= T^{''b}(\Omega \cap G_1^{-1}(C); z_0, v) \\ &= T^{''b}(\Omega; z_0, v) \cap [\nabla G_1(z_0)^{-1} \\ &\qquad \big(T^{''b}(C; G_1(z_0), \nabla G_1(z_0)v)\big)]. \end{aligned} \tag{3}$$

Proof. For each $v \in T(\Omega; z_0)$, $\nabla G_1(z_0)v \in C\big(G_1(z_0)\big)$, by Lemma 2.3 - Lemma 2.5, we have

$$\begin{aligned} T^{''}(\Omega; z_0, v) &\cap [\nabla G_1(z_0)^{-1}\big(T^{''}(C; G_1(z_0), \nabla G_1(z_0)v)\big)] \\ &= T^{''b}(\Omega; z_0, v) \cap [\nabla G_1(z_0)^{-1}\big(T^{''b}(C; G_1(z_0), \nabla G_1(z_0)v)\big)] \\ &= T^{''b}(\Omega \cap G_1^{-1}(C); z_0, v) \subset T^{''}(\Omega \cap G_1^{-1}(C); z_0, v). \end{aligned}$$

We now take any $w \in T^{''}(\Omega \cap G_1^{-1}(C); z_0, v)$. By similar arguments in the first of the proof of Lemma 2.3, we can show that

$$w \in T^{''}(\Omega; z_0, v) \cap [\nabla G_1(z_0)^{-1}\big(T^{''}(C; G_1(z_0), \nabla G_1(z_0)v)\big)].$$

Thus, the inclusion (3) is valid. □

The following lemma gives the rules for decomposition of tangent sets (cones) for the Cartesian product of sets.

Lemma 2.7. *Let* $D_i = (-\infty, 0]$ *for all* $i = 1, 2, \ldots, m$ *and* $D = \prod_{i=1}^{m} D_i$, $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m) \in D$. *Suppose that* $v = (v_1, v_2, \ldots, v_m) \in T(D; \bar{z})$. *Then*
(a) $T^2(D; \bar{z}, v) = \prod_{i=1}^{m} T^2(D_i; \bar{z}_i, v_i)$;
(b) $T^{''}(D; \bar{z}, v) = \prod_{i=1}^{m} T^{''}(D_i; \bar{z}_i, v_i)$.

Proof. By properties of the tangent cone to convex sets (see [2, page 141]), we get

$$T(D; \bar{z}) = T(\prod_{i=1}^{m} D_i; (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m)) = \prod_{i=1}^{m} T(D_i; \bar{z}_i). \tag{4}$$

Since D_i are polyhedral convex sets, we have $T^2(D_i; \bar{z}_i, v_i) = T\big(T(D_i; \bar{z}_i); v_i\big)$, for all $i = 1, 2, \ldots, m$. We get from (4) that

$$\begin{aligned} T\big(T(D; \bar{z}); v\big) &= T\big(\prod_{i=1}^{m} T(D_i; \bar{z}_i); (v_1, v_2, \ldots, v_m)\big) \\ &= \prod_{i=1}^{m} T\big(T(D_i; \bar{z}_i); v_i\big) = \prod_{i=1}^{m} T^2(D_i; \bar{z}_i, v_i). \end{aligned}$$

Hence, we have the assertion (a) of lemma.

We will prove the assertion (b) of lemma. Since D is a convex set, we get

$$T^{''}(D;\bar{z},v)=\text{cl}\Big(\text{cone}\big[\text{cone}(D-\bar{z})-v\big]\Big).$$

Combining this and the proof of [37, Lemma 5.1], we have

$$\begin{aligned}T^{''}(D;\bar{z},v)&=\text{cl}\Big(\text{cone}\big[\prod_{i=1}^{m}\text{cone}(D_i-\bar{z}_i)-(v_1,v_2,\ldots,v_m)\big]\Big)\\&=\text{cl}\Big(\text{cone}\big[\prod_{i=1}^{m}[\text{cone}(D_i-\bar{z}_i)-v_i]\big]\Big).\end{aligned}\tag{5}$$

By [37, Lemma 5.1], we get that the set

$$\text{cone}(D-\bar{z})=\text{cone}\prod_{i=1}^{m}(D_i-\bar{z}_i)=\prod_{i=1}^{m}\text{cone}(D_i-\bar{z}_i)$$

is closed and $T(D;\bar{z})=\text{cone}(D-\bar{z})$. Beside, $v=(v_1,v_2,\ldots,v_m)\in T(D;\bar{z})=\prod_{i=1}^{m}T(D_i;\bar{z}_i)$. So, $v_i\in T(D_i;\bar{z}_i)=\text{cone}(D_i-\bar{z}_i)$. Using [37, Lemma 5.1] again, we have

$$\text{cone}\big[\prod_{i=1}^{m}[\text{cone}(D_i-\bar{z}_i)-v_i]\big]=\prod_{i=1}^{m}\text{cone}[\text{cone}(D_i-\bar{z}_i)-v_i].$$

So, (5) implies

$$\begin{aligned}T^{''}(D;\bar{z},v)&=\text{cl}\Big(\prod_{i=1}^{m}\text{cone}[\text{cone}(D_i-\bar{z}_i)-v_i]\Big)\\&=\prod_{i=1}^{m}\text{cl}\Big(\text{cone}[\text{cone}(D_i-\bar{z}_i)-v_i]\Big)=\prod_{i=1}^{m}T^{''}(D_i;\bar{z}_i,v_i).\end{aligned}$$

Thus, the proof of the lemma is complete. □

The final lemma gives the rule for decomposition of the symmetric second-order subdifferential for a sum of functions in $C^{1,1}$.

Lemma 2.8. *Let D be an open subset of $\mathbb{R}^n$. Suppose that $\varphi_i:\mathbb{R}^n\to\mathbb{R}$ belong to $C^{1,1}(D)$ for all $i=1,2,\ldots,m$. Furthermore, assume that for each $j=2,3,\ldots,m$ $\nabla\varphi_j$ is strictly differentiable at $\bar{z}\in D$, with its derivative denoted by $\nabla^2\varphi_j(\bar{z})$ $\big($in particular, $\varphi_j\in C^2(D)\big)$. Then,*

$$\partial^2_S(\sum_{i=1}^{m}\varphi_i)(\bar{z})=\partial^2_S\varphi_1(\bar{z})+\sum_{i=2}^{m}\nabla^2\varphi_i(\bar{z})^*.\tag{6}$$

Proof. By the symmetric second-order subdifferential definition, we have

$$\partial^2_S\varphi_1(\bar{z})=\partial^2\varphi_1(\bar{z})\cup\partial^{+2}\varphi_1(\bar{z}),$$

with $\partial^{+2}\varphi_1(\bar{z})=D^*\partial^+\varphi_1(\bar{z})$. By [26, Corollary 4.8], we get

$$\partial^2(\varphi_1+\varphi_2)(\bar{z})=\partial^2\varphi_1(\bar{z})+\nabla^2\varphi_2(\bar{z})^*$$

and

$$\partial^{+2}(\varphi_1+\varphi_2)(\bar{z}) = \partial^{+2}\varphi_1(\bar{z}) + \nabla^2\varphi_2(\bar{z})^*.$$

So,

$$\partial^2_S(\varphi_1+\varphi_2)(\bar{z}) = \partial^2_S\varphi_1(\bar{z}) + \nabla^2\varphi_2(\bar{z})^*.$$

Hence, the equality (6) is satisfied for $m = 2$. Assume that the equality (6) is satisfied for $m = n-1$, we will prove that it also holds for $m = n$. By the assumptions of lemma, the function $\sum_{i=1}^{n-1}\varphi_i$ belongs to $C^{1.1}(D)$. From [26, Corollary 4.8], we have

$$\partial^2_S(\sum_{i=1}^{n}\varphi_i)(\bar{z}) = \partial^2_S\Big(\sum_{i=1}^{n-1}\varphi_i + \varphi_n\Big)(\bar{z}) = \partial^2_S\big(\sum_{i=1}^{n-1}\varphi_i\big)(\bar{z}) + \nabla^2\varphi_n(\bar{z})^*.$$

Combining this and the inductive hypothesis, we obtain

$$\partial^2_S(\sum_{i=1}^{n}\varphi_i)(\bar{z}) = \partial^2_S\big(\sum_{i=1}^{n-1}\varphi_i\big)(\bar{z}) + \nabla^2\varphi_n(\bar{z})^* = \partial^2_S\varphi_1(\bar{z}) + \sum_{i=2}^{m}\nabla^2\varphi_i(\bar{z})^*.$$

The proof of the lemma is complete. □

3. Optimality Conditions for Mathematical Programming Problems

In this section, we suppose that Z, E, Y are finite-dimensional Euclidean spaces, $f_1 : Z \to \mathbb{R}$ is a given function. Assume moreover that two functions $\mathcal{F} : Z \to Y$, $\mathcal{H} : Z \to E$ have second-order Gâteaux derivatives around $z_0 \in Z$, and $\mathcal{D} \subset Y$ is a nonempty closed convex set. Let $\mathcal{H}^* : E \to Z$ be an adjoint mapping of $\mathcal{H}$. We put

$$\mathcal{A} = \{z \in Z : \mathcal{H}(z) = 0\}.$$

We are interested in deriving the second-order optimality conditions for the following problem

$$(P) \qquad\qquad \text{Minimize}\{f_1(z) : z \in \mathcal{A}, \text{ and } \mathcal{F}(z) \in \mathcal{D}\}$$

by using the asymptotic second-order tangent cone.

3.1. Second-Order Necessary Optimality Conditions

We now establish our first main result: the second-order necessary conditions for the problem (P) under the assumption that $f_1 \in C^{1,1}(Z)$.

Theorem 3.1. *Suppose $z_0 \in \mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D})$ is a local minimum for (P) at which the regularity condition (1) is satisfied, $T(\mathcal{D}; \mathcal{F}(z_0)) = \mathcal{D}\big(\mathcal{F}(z_0)\big)$, $\nabla\mathcal{H}(\cdot)$ is continuous at z_0, and $\nabla\mathcal{H}(z_0)$ is surjective. Then, the following assertions hold:*

(a) $\langle\nabla f_1(z_0), v\rangle \geq 0$ for all $v \in Z : \nabla\mathcal{H}(z_0)v = 0$ and $\nabla\mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$;

(b) For any $v \in Z$, $\nabla \mathcal{H}(z_0)v = 0$, $\nabla \mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$, $\langle \nabla f_1(z_0), v\rangle = 0$, one has

$$\langle \nabla f_1(z_0), u\rangle \geq 0 \ \forall u \in v^{\perp}, \ \ \nabla \mathcal{H}(z_0)u = 0, \ \nabla \mathcal{F}(z_0)u \in T^{''}\big(\mathcal{D}; \mathcal{F}(z_0), \nabla \mathcal{F}(z_0)v\big)$$

and there is $z^* \in \partial^2_S f_1(z_0)(v)$ such that

$$\langle \nabla f_1(z_0), w\rangle + \langle z^*, v\rangle \geq 0, \forall w \in v^{\perp}, \text{ and } \nabla \mathcal{H}(z_0)w + \nabla^2 \mathcal{H}(z_0)vv = 0,$$
$$\nabla \mathcal{F}(z_0)w \in T^2\big(\mathcal{D}; \mathcal{F}(z_0), \nabla \mathcal{F}(z_0)v - \nabla^2 \mathcal{F}(z_0)vv\big).$$

Proof. For all $v \in Z$ such that $\nabla \mathcal{H}(z_0)v = 0$ and $\nabla \mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$, we have from [18, Lemma 2.2 (i)] that $v \in T(\mathcal{A}; z_0) = T^b(\mathcal{A}; z_0)$ and

$$v \in \nabla \mathcal{F}(z_0)^{-1}\big(T(\mathcal{D}; \mathcal{F}(z_0))\big) = \mathcal{F}(z_0)^{-1}\big(T^b(\mathcal{D}; \mathcal{F}(z_0))\big).$$

By [18, Theorem 2.3 (i)],

$$\begin{aligned} v \in T^b(\mathcal{A}; z_0) \cap \nabla \mathcal{F}(z_0)^{-1}\big(T^b(\mathcal{D}; \mathcal{F}(z_0))\big) &= T^b\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0\big) \\ &\subset T\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0\big). \end{aligned}$$

Since [10, Corollary 4.2], we have the assertion (a). We will prove the assertion (b). For any $v \in Z$, $\nabla \mathcal{H}(z_0)v = 0$, $\nabla \mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$ and $\langle \nabla f_1(z_0), v\rangle = 0$ we get $v \in T\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0\big)$. So

$$v \in T\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0\big) \cap \ker \nabla f_1(z_0).$$

For any $u \in v^{\perp}$ and $\nabla \mathcal{H}(z_0)u = 0$, $\nabla \mathcal{F}(z_0)u \in T^{''}\big(\mathcal{D}; \mathcal{F}(z_0), \nabla \mathcal{F}(z_0)v\big)$, we have from Lemma 2.4 and Lemma 2.5 that

$$u \in T(\mathcal{A}; z_0) = T^{''}(\mathcal{A}; z_0, v) = T^{''b}(\mathcal{A}; z_0, v)$$

and

$$u \in \nabla \mathcal{F}(z_0)^{-1}\big(T^{''b}(\mathcal{D}; \mathcal{F}(z_0), \nabla \mathcal{F}(z_0)v)\big).$$

By Corollary 2.6,

$$\begin{aligned} u &\in T^{''b}(\mathcal{A}; z_0, v) \cap \nabla \mathcal{F}(z_0)^{-1}\big(T^{''b}(\mathcal{D}; \mathcal{F}(z_0), \nabla \mathcal{F}(z_0)v)\big) \\ &= T^{''b}\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0, v\big) \\ &= T^{''}\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0, v\big). \end{aligned}$$

So,

$$u \in T^{''}\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0, v\big) \cap v^{\perp}.$$

Then, there exist $(t_k, r_k) \to (0^+, 0^+)$, $u_k \to u$ such that $\frac{t_k}{r_k} \to 0$ and

$$z_k = z_0 + t_k v + \frac{1}{2} t_k r_k u_k \in \mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}), \text{ for all } k \in \mathbb{N}.$$

From $\lim\limits_{k\to\infty} z_k = z_0$ and z_0 is a local minimum for (P), we have

$$f_1(z_k) - f_1(z_0) \geq 0, \text{ for all large enough } k.$$

Since $\langle \nabla f_1(z_0), v\rangle = 0$, we get

$$f_1(z_k)-f_1(z_0) = [f_1(z_k)-f_1(z_0+t_kv)]+[f_1(z_0+t_kv)-f_1(z_0)-\langle \nabla f_1(z_0), t_kv\rangle].$$

By [10, Corollary 2.1], there exist $\theta_k \in (0,1)$ and $w_k^* \in \partial_S^2 f_1(z_0+\theta_k t_k v)(t_kv)$ such that

$$f_1(z_0+t_kv) - f_1(z_0) - \langle \nabla f_1(z_0), t_kv\rangle = \frac{1}{2}\langle w_k^*, t_kv\rangle = \frac{1}{2}t_k\langle w_k^*, v\rangle.$$

Note that $\partial_S^2 f_1(z_0+\theta_k t_k v)(t_kv) = t_k\partial_S^2 f_1(z_0+\theta_k t_k v)(v)$. So, there is

$$z_k^* \in \partial_S^2 f_1(z_0+\theta_k t_k v)(v)$$

such that $w_k^* = t_k z_k^*$. Hence,

$$f_1(z_0+t_kv) - f_1(z_0) - \langle \nabla f_1(z_0), t_kv\rangle = \frac{1}{2}\langle w_k^*, t_kv\rangle = \frac{1}{2}t_k^2\langle z_k^*, v\rangle. \qquad (7)$$

By the mean value theorem, there exists ε_k between z_0+t_kv and z_k such that

$$f_1(z_k) - f_1(z_0+t_kv) = \langle \nabla f_1(\varepsilon_k), \frac{1}{2}t_kr_ku_k\rangle.$$

Combining this with (7), we have

$$f_1(z_k) - f_1(z_0) = \frac{1}{2}t_k^2\langle z_k^*, v\rangle + \frac{1}{2}t_kr_k\langle \nabla f_1(\varepsilon_k), u_k\rangle \geq 0, \qquad (8)$$

for all large enough k. This implies that

$$\frac{t_k}{r_k}\langle z_k^*, v\rangle + \langle \nabla f_1(\varepsilon_k), u_k\rangle \geq 0,$$

for all large enough k. Taking $k \to \infty$, we obtain

$$\langle \nabla f_1(z_0), u\rangle \geq 0.$$

We will prove the second part. Take any $w \in v^\perp$ and

$$\nabla\mathcal{H}(z_0)w + \nabla^2\mathcal{H}(z_0)vv = 0,\ \nabla\mathcal{F}(z_0)w \in T^2\big(\mathcal{D}; \mathcal{F}(z_0), \nabla\mathcal{F}(z_0)v - \nabla^2\mathcal{F}(z_0)vv\big).$$

We have $w \in T^2(\mathcal{A}; z_0, v) = T^{2b}(\mathcal{A}; z_0, v)$ and $w \in \nabla\mathcal{F}(z_0)^{-1}$ $\Big(T^2\big(\mathcal{D}; \mathcal{F}(z_0), \nabla\mathcal{F}(z_0)v - \nabla^2\mathcal{F}(z_0)vv\big)\Big)$ by [18, Lemma 2.2 (ii)]. Since [17, Theorem 4.9.5],

$$\nabla\mathcal{F}(z_0)v - \nabla^2\mathcal{F}(z_0)vv \in T\big(\mathcal{D}; \mathcal{F}(z_0)\big) = \mathcal{D}\big(\mathcal{F}(z_0)\big).$$

So, $T^2\big(\mathcal{D}; \mathcal{F}(z_0), \nabla\mathcal{F}(z_0)v - \nabla^2\mathcal{F}(z_0)vv\big) = T^{2b}\big(\mathcal{D}; \mathcal{F}(z_0), \nabla\mathcal{F}(z_0)v - \nabla^2\mathcal{F}(z_0)vv\big)$ from [18, Lemma 2.3]. By similar arguments in the proof of Corollary 2.6, we can show that

$$w \in T^{2b}\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0, v\big) = T^2\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0, v\big).$$

So, there exist $t'_k \to 0^+$, $w_k \to w$ such that

$$z'_k = z_0 + t'_kv + \frac{1}{2}(t'_k)^2w_k \in \mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}),\ \text{for all } k \in \mathbb{N}.$$

From $\lim_{k\to\infty} z'_k = z_0$ and z_0 is a local minimum for (P), we have

$$f_1(z'_k) - f_1(z_0) \geq 0, \text{ for all large enough } k. \tag{9}$$

Since $\langle \nabla f_1(z_0), v\rangle = 0$, we get

$$f_1(z'_k)-f_1(z_0) = [f_1(z'_k)-f_1(z_0+t'_kv)]+[f_1(z_0+t'_kv)-f_1(z_0)-\langle \nabla f_1(z_0), t'_kv\rangle]. \tag{10}$$

Similar to the proof of formula (7), we can show that there exist $\theta'_k \in (0,1)$ and

$$y^*_k \in \partial^2_S f_1(z_0 + \theta'_k t'_k v)(v)$$

such that

$$f_1(z_0 + t'_kv) - f_1(z_0) - \langle \nabla f_1(z_0), t'_kv\rangle = \frac{1}{2}(t'_k)^2\langle y^*_k, v\rangle, \tag{11}$$

for all $k \in \mathbb{N}$. From the mean value theorem, there is γ_k between $z_0 + t'_kv$ and z'_k such that

$$f_1(z'_k) - f_1(z_0 + t'_kv) = \langle \nabla f_1(\gamma_k), \frac{1}{2}(t'_n)^2 w_k\rangle = \frac{1}{2}(t'_n)^2\langle \nabla f_1(\gamma_k), w_k\rangle, \tag{12}$$

for all $k \in \mathbb{N}$. Combining (11), (12) and (10), we have

$$f_1(z'_k) - f_1(z_0) = \frac{1}{2}(t'_k)^2\langle y^*_k, v\rangle + \frac{1}{2}(t'_n)^2\langle \nabla f_1(\gamma_k), w_k\rangle, \tag{13}$$

for all large enough k. This and (9) imply that

$$\langle y^*_k, v\rangle + \langle \nabla f_1(\gamma_k), w_k\rangle \geq 0, \tag{14}$$

for all large enough k. By Proposition 2.3 (iii), $\partial^2_S f_1(\cdot)(v)$ is locally bounded at z_0. Moreover, $\lim_{k\to\infty}(z_0 + \theta'_k t'_k v) = z_0$, we have that $\{y^*_k\}$ is bounded. So, we can assume that $\lim_{k\to\infty} y^*_k = z^* \in \partial^2_S f_1(z_0)(v)$. Passing $k \to \infty$ in (14), we obtain

$$\langle \nabla f_1(z_0), w\rangle + \langle z^*, v\rangle \geq 0.$$

Thus, the proof of the theorem is complete. □

Remark 3.2. Compared to Huy and Tuyen's results [10], which derived second-order necessary optimality conditions for mathematical programming problems with only geometric constraints, our work extends their results. Besides, when the objective functions are not of class C^2, Toan and Thuy [39] have only derived directional *first-order* necessary optimality conditions for the problem (P). Theorem 3.1 improves the result in [39, Theorem 1] by establishing *second-order* necessary optimality conditions for problem (P), getting closer to the sufficient optimality conditions presented in the following subsection.

3.2. Second-Order Sufficient Optimality Conditions

Definition 3.3. (see [41, Definition 1.1]) A point z_0 is said to be a *strict local minimizer of order* 2 for (P), iff there exist $\delta > 0$ and $\rho > 0$ such that:

$$f_1(x) > f_1(z_0) + \rho\|z - z_0\|^2,\ \forall z \in \mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}) \cap [B(z_0, \delta)\backslash\{z_0\}].$$

The following theorem provides sufficient conditions for a strict local minimizer of (P), constituting our second main result.

Theorem 3.4. *Suppose* $z_0 \in \mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D})$, $T(\mathcal{D}; \mathcal{F}(z_0)) = \mathcal{D}\big(\mathcal{F}(z_0)\big)$, $\nabla\mathcal{H}(\cdot)$ *is continuous at* z_0, $\nabla\mathcal{H}(z_0)$ *is surjective, and the regularity condition* (1) *is satisfied. Assume moreover that the following conditions are hold:*
(a) $\langle\nabla f_1(z_0), v\rangle \geq 0$, for all $v \in Z : \nabla\mathcal{H}(z_0)v = 0$ and $\nabla\mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$;
(b) For any $v \in Z\backslash\{0\}$, $\nabla\mathcal{H}(z_0)v = 0$, $\nabla\mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$, $\langle\nabla f_1(z_0), v\rangle = 0$, *one has*

$$\langle\nabla f_1(z_0), u\rangle > 0\ \forall u \in v^{\perp}\backslash\{0\}, \qquad \nabla\mathcal{H}(z_0)u = 0,\ \nabla\mathcal{F}(z_0)u \in T^{''}\big(\mathcal{D}; \mathcal{F}(z_0), \nabla\mathcal{F}(z_0)v\big) \tag{15}$$

and

$$\begin{aligned}&\langle\nabla f_1(z_0), w\rangle + \langle z^*, v\rangle > 0,\ \forall w \in v^{\perp},\quad \nabla\mathcal{H}(z_0)w + \nabla^2\mathcal{H}(z_0)vv = 0,\\ &\quad \nabla\mathcal{F}(z_0)w \in T^2\big(\mathcal{D}; \mathcal{F}(z_0), \nabla\mathcal{F}(z_0)v - \nabla^2\mathcal{F}(z_0)vv\big), \forall z^* \in \partial_S^2 f_1(z_0)(v).\end{aligned} \tag{16}$$

Then, z_0 *is a strict local minimizer of order* 2 *for* (P).

We will prove theorem by contradiction. Suppose that z_0 is not a strict local minimizer of order 2 for (P). Then, for each $\rho_k \to 0^+$, there exist $z_k \in \mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D})\backslash\{z_0\}$ such that $\lim_{k\to\infty} z_k = z_0$ and

$$f_1(z_k) - f_1(z_0) \leq \rho_k\|z_k - z_0\|^2,\ \forall k \in \mathbb{N}. \tag{17}$$

Put $t_k = \|z_k - z_0\|$ and $v_k = \frac{z_k - z_0}{t_k}$, for all $k \in \mathbb{N}$. We get $\|v_k\| = 1$, for all $k \in \mathbb{N}$. Without any loss of generality, we can suppose that $\lim_{k\to\infty} v_k = v$. Note that $z_k = z_0 + t_k v_k$ and $t_k = \|z_k - z_0\| \to 0$, as $k \to \infty$. So, $v \in T\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0\big)$ and $\|v\| = 1$. By similar arguments in the first of the proof of Lemma 2.3, we can show that $v \in T\big(\mathcal{A}; z_0\big)$ and $\nabla\mathcal{F}(z_0)v \in T\big(\mathcal{D}; \mathcal{F}(z_0)\big)$. From Lemma 2.4, we have $\nabla\mathcal{H}(z_0)v = 0$ and $\nabla\mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$. From the Fréchet differentiability of f_1 and (17), we have

$$\langle\nabla f_1(z_0), v\rangle = \lim_{k\to\infty} \frac{f_1(z_k) - f_1(z_0)}{t_k} \leq \lim_{k\to\infty} \rho_k t_k = 0.$$

Combining this with (a), we get $\langle\nabla f_1(z_0), v\rangle = 0$. Thus, $v \in Z\backslash\{0\}$, $\nabla\mathcal{H}(z_0)v = 0$, $\nabla\mathcal{F}(z_0)v \in T(\mathcal{D}; \mathcal{F}(z_0))$, $\langle\nabla f_1(z_0), v\rangle = 0$. On the other hand, since [15, Lemma 3.4, pages 129-130], we can assume that there is a subsequence of $\{z_k\}$ denoted also by $\{z_k\}$, such that either

(i) there is a sequence $r_k \to 0^+$ such that $\frac{t_k}{r_k} \to 0$ and $u_k := \frac{z_k - z_0 - t_k v}{\frac{1}{2} t_k r_k}$ converges to some vector $u \in T^{''}\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0, v\big) \cap v^{\perp} \backslash \{0\}$ or
(ii) $w_k := \frac{z_k - z_0 - t_k v}{\frac{1}{2} t_k^2}$ converges to some vector $w \in T^2\big(\mathcal{A} \cap \mathcal{F}^{-1}(\mathcal{D}); z_0, v\big) \cap v^{\perp}$.

For (i), we get $z_k = z_0 + t_k v + \frac{1}{2} t_k r_k u_k$. Note that $\langle \nabla f_1(z_0), v \rangle = 0$. Similar to the proof of formula (8) in Theorem 3.1, we can show that there exist $\theta_k \in (0, 1)$, $z_k^* \in \partial_S^2 f_1(z_0 + \theta_k t_k v)(v)$ and ε_k between $z_0 + t_k v$ and z_k such that

$$f_1(z_k) - f_1(z_0) = \frac{1}{2} t_k^2 \langle z_k^*, v \rangle + \frac{1}{2} t_n r_n \langle \nabla f_1(\varepsilon_k), u_k \rangle, \tag{18}$$

for all large enough k. Substituting (18) into the left side of (17) we obtain

$$\frac{1}{2} t_k r_k \langle \nabla f_1(\varepsilon_k), u_k \rangle + \frac{1}{2} t_k^2 \langle z_k^*, v \rangle \leq \rho_k \| z_k - z_0 \|^2, \text{ for all large enough } k.$$

So,

$$\langle \nabla f_1(\xi_k), u_k \rangle + \frac{t_k}{r_k} \langle z_k^*, v \rangle \leq 2 \frac{t_k}{r_k} \rho_k \| v + \frac{1}{2} t_k u_k \|^2, \text{ for all large enough } k.$$

Similar to the proof at the end of Theorem 3.1, we can assume that $\lim_{k \to \infty} z_k^* = z^* \in \partial_S^2 f_1(z_0)(v)$. Taking $k \to \infty$ and note that $\frac{t_k}{r_k} \to 0$, $\rho_k \to 0^+$, we obtain $\langle \nabla f_1(z_0), u \rangle \leq 0$, this is contradictable with (15). Hence, the assertion (a) is proven.

For (ii), we get $z_k = z_0 + t_k v + \frac{1}{2} t_k^2 w_k$. Similar to the proof of formula (13) in Theorem 3.1, we can show that there exist $\theta_k \in (0, 1)$, $z_k^* \in \partial_S^2 f_1(z_0 + \theta_k t_k v)(v)$ and ξ_k between $z_0 + t_k v$ and z_k such that

$$f_1(z_k) - f_1(z_0) = \frac{1}{2} t_k^2 \langle \nabla f_1(\xi_k), w_k \rangle + \frac{1}{2} t_k^2 \langle z_k^*, v \rangle, \tag{19}$$

for all large enough k. Combining (17) and (19), we have

$$\frac{1}{2} t_k^2 \langle \nabla f_1(\xi_k), w_k \rangle + \frac{1}{2} t_k^2 \langle z_k^*, v \rangle \leq \rho_k \| z_k - z_0 \|^2, \text{ for all large enough } k.$$

So,

$$\langle \nabla f_1(\xi_k), w_k \rangle + \langle z_k^*, v \rangle \leq 2 \rho_k \| v + \frac{1}{2} t_k w_k \|^2, \text{ for all large enough } k.$$

Similar to the proof at the end of Theorem 3.1, we can also assume that $\lim_{k \to \infty} z_k^* = z^* \in \partial_S^2 f_1(z_0)(v)$. Taking $k \to \infty$, we obtain $\langle \nabla f_1(z_0), w \rangle + \langle z^*, v \rangle \leq 0$, this is contradictable with (16). Hence, the assertion (b) is also proven. Thus, z_0 is a strict local minimizer of order 2 for (P). The proof of the theorem is complete.

Remark 3.5. When the objective functions are of class C^2, the authors in [36] employ a stricter regularity condition than that in [37] to derive sufficient optimality conditions for problem (P). In contrast, Theorem 3.4 derives second-order sufficient optimality conditions using the same regularity condition as in [37,39]. Furthermore, we achieve a "no-gap" property between the second-order necessary and sufficient optimality conditions in this case.

4. Example and Application

4.1. An Example

We now give an example to illustrate our main results.

Example 4.1. Let $Z = Y = \mathbb{R}^3,\ E = \mathbb{R},\ \mathcal{D} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$,

$$\mathcal{H}(z) = \mathcal{H}(z_1, z_2, z_3) = z_1^2 + \sin z_2 - 2z_2 + 2z_3^2 \text{ and}$$
$$\mathcal{F}(z) = \mathcal{F}(z_1, z_2, z_3) = (z_1, z_2, z_3).$$

We consider the following optimization problem

$$\text{Min}\{f_1(z) : \mathcal{H}(z) = 0,\ \mathcal{F}(z) \in \mathcal{D}\}, \tag{20}$$

where

$$f_1(z) = f_1(z_1, z_2, z_3) = \begin{cases} 3z_1^2 + 2z_2^2 + z_2 + 2z_3^2 + z_3 + z_1^2 \cos(\ln |z_1|) & \text{if } z_1 \neq 0 \\ 2z_2^2 + z_2 + 2z_3^2 + z_3 & \text{if } z_1 = 0. \end{cases}$$

Then, $\bar{z} = (0, 0, 0)$ is a strict local minimum of order 2 for problem (20).

Indeed, it is easy to check that $\nabla f_1(\bar{z}) = (0, 1, 1)^T$, $f \in C^{1,1}(\mathbb{R}^3)$, $\nabla \mathcal{H}(\cdot)$ is continuous at $\bar{z}$ and $\nabla \mathcal{H}(\bar{z})$ is surjective. We now prove that the regularity condition (1) is also satisfied. For each $r > 0$, $\hat{z} = (\hat{z}_1, \hat{z}_2, \hat{z}_3) \in B(\bar{z}, r) \cap \mathcal{A}$, we have

$$\nabla \mathcal{F}(\bar{z})\big(T(\mathcal{A}; \hat{z})\big) = T\big(\mathcal{A}; \hat{z}\big) = \big\{(w_1, w_2, w_3) \in \mathbb{R}^3 : 2\hat{z}_1 w_1 + (\cos \hat{z}_2 - 2)w_2 + 4\hat{z}_3 w_3 = 0\big\}.$$

Note that $\mathcal{D}\big(\mathcal{F}(\bar{z})\big) = \mathcal{D}(\bar{z}) = \mathcal{D}$. We will prove that

$$T\big(\mathcal{A}; \hat{z}\big) - \mathcal{D} = \mathbb{R}^3.$$

Take any $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. We can assume that $y_3 = y_3^1 - y_3^2$, with $y_3^2 \geq 0$. Choose $w_1 = 0, w_3 = y_3^1$. There exists $w_2 \in \mathbb{R}$ such that $(\cos \hat{z}_2 - 2)w_2 = -4\hat{z}_3 w_3 = -4\hat{z}_3 y_3^1$. Then, $w = (w_1, w_2, w_3) = (0, w_2, y_3^1) \in T\big(\mathcal{A}; \hat{z}\big)$, $(-y_1, -y_2 + w_2, y_3^2) \in \mathcal{D}$ and

$$(0, w_2, y_3^1) - (-y_1, -y_2 + w_2, y_3^2) = (y_1, y_2, y_3).$$

Thus, $T\big(\mathcal{A}; \hat{z}\big) - \mathcal{D} = \mathbb{R}^3$. We have

$$T\big(\mathcal{D}; \mathcal{F}(\bar{z})\big) = T\big(\mathcal{D}; \bar{z}\big) = \mathcal{D}$$

and

$$T^2\big(\mathcal{D};\mathcal{F}(\bar z),v\big)=T''\big(\mathcal{D};\mathcal{F}(\bar z),v\big)=T\big(T\big(\mathcal{D};\mathcal{F}(\bar z)\big);v\big)=T(\mathcal{D};v)$$
$$=\begin{cases}\mathcal{D} & \text{if } v=(v_1,v_1,0)\\ \mathbb{R}^3 & \text{if } v=(v_1,v_1,v_3),\ v_3>0.\end{cases}$$

Take any $v=(v_1,v_1,v_3)\in Z$ such that $\nabla\mathcal{H}(\bar z)v=0$ and $\nabla\mathcal{F}(\bar z)v\in T(\mathcal{D};\mathcal{F}(\bar z))$, we have $(v_1,v_1,v_3)\in T(\mathcal{D};\mathcal{F}(\bar z))=\mathcal{D}$. Note that $\nabla\mathcal{H}(\bar z)v=-v_2$. So, $v_3\geq 0$ and $v_2=0$. Hence, $\langle\nabla f_1(\bar z),v\rangle=v_2+v_3\geq 0$, and so the assertion (a) of Theorem 3.4 is satisfied. Take any $v=(v_1,v_1,v_3)\in Z\backslash\{0\}$, $\nabla\mathcal{H}(\bar z)v=0$, $\nabla\mathcal{F}(\bar z)v\in T(\mathcal{D};\mathcal{F}(\bar z))$, $\langle\nabla f_1(\bar z),v\rangle=0$, we have $-v_2=0,\ v_3\geq 0$, $v_2+v_3=0$. So, $v_2=v_3=0,\ v_1\neq 0$. Hence, $v=(v_1,0,0)$ with $v_1\neq 0$. We now take $u\in v^\perp\backslash\{0\},\ \nabla\mathcal{H}(\bar z)u=0,\ \nabla\mathcal{F}(\bar z)u\in T''\big(\mathcal{D};\mathcal{F}(\bar z),\nabla\mathcal{F}(\bar z)v\big)$, we get $u=(0,u_2,u_3),\ -u_2=0,\ \nabla\mathcal{F}(\bar z)u=u=(0,0,u_3)\in T''\big(\mathcal{D};\mathcal{F}(\bar z),\nabla\mathcal{F}(\bar z)v\big)=T''\big(\mathcal{D};\bar z,v\big)=\mathcal{D}$. So, $u_3>0$. Hence,

$$\langle\nabla f_1(\bar z),u\rangle=u_3>0,$$

and so (15) is satisfied. We will check for (16). Take any $w\in v^\perp,\ z^*\in\partial_S^2 f_1(\bar z)(v)$ such that $\nabla\mathcal{H}(\bar z)w+\nabla^2\mathcal{H}(\bar z)vv=0,\ \nabla\mathcal{F}(\bar z)w\in T^2\big(\mathcal{D};\mathcal{F}(\bar z),\nabla\mathcal{F}(\bar z)v-\nabla^2\mathcal{F}(\bar z)vv\big)$, we get $w=(0,w_2,w_3),\ -w_2+2v_1^2+4v_2^2=-w_2+2v_1^2=0,\ \nabla\mathcal{F}(\bar z)w=(0,w_2,w_3)\in T^2\big(\mathcal{D};\mathcal{F}(\bar z),\nabla\mathcal{F}(\bar z)v-\nabla^2\mathcal{F}(\bar z)vv\big)=T^2\big(\mathcal{D};\bar z,v\big)=\mathcal{D}$. So, $w_2=2v_1^2>0$ and $w_3\geq 0$. Hence, $\langle\nabla f_1(\bar z),w\rangle=w_2+w_3>0$. Note that $v=(v_1,0,0)$, we have

$$\partial_S^2 f_1(\bar z)(v)=\partial_S\langle v,\nabla f_1(\cdot)\rangle(\bar z)=\partial_S\big(v_1\nabla_{z_1}f_1(\cdot)\big)(\bar z)$$
$$=v_1\partial_S\big(\nabla_{z_1}f_1(\cdot)\big)(\bar z)=\big(v_1\partial_S g(0),0,0\big),$$

where

$$g(z_1):=\nabla_{z_1}f(z)=6z_1+2z_1\cos(\ln|z_1|)-z_1\sin(\ln|z_1|),$$
$$\text{for all } z=(z_1,z_2,z_3)\in Z.$$

From the Lipschitzian property of $g(\cdot)$, one has

$$\partial_S g(0)=\Big[\liminf_{z_1\to 0,z_1\neq 0}\nabla g(z_1),\ \limsup_{z_1\to 0,z_1\neq 0}\nabla g(z_1)\Big].$$

We have

$$\nabla g(z_1)=6+\cos(\ln|z_1|)-3\sin(\ln|z_1|),\ \forall z_1\neq 0.$$

So, we can check that

$$\partial_S^2 f_1(\bar z)(v)=\Big\{z^*=(z_1^*,z_2^*,z_3^*)\in Z: z_1^*=av_1, z_2^*=z_3^*=0,$$
$$a\in[6-\sqrt{10},6+\sqrt{10}\Big\}.$$

Hence,

$$\langle \nabla f_1(\bar{z}), w\rangle + \langle z^*, v\rangle = w_2 + w_3 + av_1^2 > 0,$$

for all $z^* = (av_1, 0, 0) \in \partial_S^2 f_1(\bar{z})(v)$. Thus, the assertion (b) of Theorem 3.4 is also satisfied. By Theorem 3.4, $\bar{z} = (0, 0, 0)$ is a strict local minimum of order 2 for problem (20).

4.2. Application to an optimal control problem

Let X_k, U_k for each $k = 0, 1, ..., N-1$, and X_N are the finite-dimensional Euclidean spaces, where N is a positive natural number. Assume that

- $h_k : X_k \times U_k \to \mathbb{R}$, $h_N : X_N \to \mathbb{R}$ are real functions;
- $\varphi_k : X_k \times U_k \to X_{k+1}$ is a vector function;
- $g_k : X_k \times U_k \to \mathbb{R}$ is a real function.

We describe a control system with state variable x_k and control variable u_k at time k. The objective function is the sum of functions h_k for $k = 0, 1, \ldots, N$. The state variable space at stage k is denoted by X_k, and the control variable space at stage k is denoted by U_k.

Consider the following discrete optimal control problem: Find a pair (x, u) where $x = (x_0, x_1, \ldots, x_N) \in X_0 \times X_1 \times \cdots \times X_N$ is a trajectory and $u = (u_0, u_1, \ldots, u_{N-1}) \in U_0 \times U_1 \times \cdots \times U_{N-1}$ is a control sequence, which minimizes the objective function

$$\sum_{k=0}^{N-1} h_k(x_k, u_k) + h_N(x_N), \tag{21}$$

and satisfy the state equation

$$x_{k+1} = \varphi_k(x_k, u_k), \quad k = 0, 1, \ldots, N-1, \tag{22}$$

the initial condition

$$x_0 = c \in X_0, \tag{23}$$

and the constraints

$$g_k(x_k, u_k) \leq 0, \quad k = 0, 1, \ldots, N-1. \tag{24}$$

This class of problems has been studied extensively in the literature; see, for example, [1,3,7,19–22,24,25,33,35–37,40] and the references therein for further details.

Define the product spaces:

$$X = X_0 \times X_1 \times \cdots \times X_N, \quad U = U_0 \times U_1 \times \cdots \times U_{N-1}.$$

For each $x = (x_0, x_1, \ldots, x_N) \in X$ and $u = (u_0, u_1, \ldots, u_{N-1}) \in U$, we put

$$f(x, u) = \sum_{k=0}^{N-1} h_k(x_k, u_k) + h_N(x_N)$$

and

$$F(x,u)=\big(g_0(x_0,u_0),g_1(x_1,u_1),\ldots,g_{N-1}(x_{N-1},u_{N-1})\big). \tag{25}$$

Let

$$D_k=(-\infty,0]\quad(k=0,1,\ldots,N-1),\quad D=\prod_{k=0}^{N-1}D_k,\ Z=X\times U.$$

Then, problem (21)–(24) can be written as the following form:

$$\begin{aligned}&\text{Minimize}\ \ f(z)\\&\text{subject to}\ \ z\in A,\quad F(z)\in D,\end{aligned} \tag{26}$$

where $H:Z\to X$ is defined by

$$\begin{aligned}H(z)&=H(x_0,x_1,\ldots,x_N,u_0,u_1,\ldots,u_{N-1})\\&=\big(x_0-c,x_1-\varphi_0(x_0,u_0),x_2-\varphi_1(x_1,u_1),\ldots,x_N-\varphi_{N-1}(x_{N-1},u_{N-1})\big)\end{aligned}$$

and

$$A=\{z\in Z:H(z)=0\}. \tag{27}$$

A pair $(\bar{x},\bar{u})$ satisfying (22), (23), and (24) is called *admissible* for the problem (21)–(24). For a given admissible couple $(\bar{x},\bar{u})$, symbols $\bar{h}_k,\frac{\partial\bar{h}_k}{\partial u_k},\frac{\partial^2\bar{h}_k}{\partial u_k\partial x_k}$, etc., stand, respectively, for $h_k(\bar{x}_k,\bar{u}_k),(\frac{\partial h_k}{\partial u_k})(\bar{x}_k,\bar{u}_k)$, $(\frac{\partial^2 h_k}{\partial u_k\partial x_k})(\bar{x}_k,\bar{u}_k)$, etc., where

$$\frac{\partial^2\bar{h}_k}{\partial x_k\partial u_k}=\begin{bmatrix}\frac{\partial^2\bar{h}_k}{\partial x_k^1\partial u_k^1}&\frac{\partial^2\bar{h}_k}{\partial x_k^1\partial u_k^2}&\cdots&\frac{\partial^2\bar{h}_k}{\partial x_k^1\partial u_k^M}\\\frac{\partial^2\bar{h}_k}{\partial x_k^2\partial u_k^1}&\frac{\partial^2\bar{h}_k}{\partial x_k^2\partial u_k^2}&\cdots&\frac{\partial^2\bar{h}_k}{\partial x_k^2\partial u_k^M}\\\vdots&\vdots&\vdots&\vdots\\\frac{\partial^2\bar{h}_k}{\partial x_k^S\partial u_k^1}&\frac{\partial^2\bar{h}_k}{\partial x_k^S\partial u_k^2}&\cdots&\frac{\partial^2\bar{h}_k}{\partial x_k^S\partial u_k^M}\end{bmatrix}$$

if $x_k=(x_k^1,x_k^2,\ldots,x_k^S)\in X_k=\mathbb{R}^S$, and $u_k=(u_k^1,u_k^2,\ldots,u_k^M)\in U_k=\mathbb{R}^M$. We denote x_k^T as the transpose of the matrix x_k. An admissible couple $(\bar{x},\bar{u})$ is said to be a *local minimum* for the problem (21)–(24) if there exists $\epsilon>0$ such that for all admissible couples (x,u), the following implication holds:

$$\|(x,u)-(\bar{x},\bar{u})\|_Z\le\epsilon\Rightarrow f(x,u)\ge f(\bar{x},\bar{u}).$$

We now state assumptions for the problem (21)–(24).

(H_1) The function h_0 is Fréchet differentiable around $(\bar{x}_0,\bar{u}_0)$, whose gradient mapping ∇h_0 is locally Lipschitz around $(\bar{x}_0,\bar{u}_0)$. For each $k=1,2,\ldots,N-1$, the function h_k belongs to C^2 around $(\bar{x}_k,\bar{u}_k)$, and the function h_N belongs to C^2 around $\bar{x}_N$.

(H_2) For each $k=0,1,\ldots,N-1$, the functions φ_k and g_k are twice differentiable around $(\bar{x}_k,\bar{u}_k)$.

(H_3) For all $(\breve{x}, \breve{u})$ in neighborhood of $(\bar{x}, \bar{u})$, for all $k \in I(\bar{x}, \bar{u})$ and $v_k \leq 0$, there exist $x_0 = 0,\quad u_l \in U_l\ (l = 0, 1, \ldots, N-1)$ such that

$$\frac{\partial \bar{g}_k}{\partial x_k} x_k + \frac{\partial \bar{g}_k}{\partial u_k} u_k - v_k \leq 0 \quad \text{if } k \in I(\bar{x}, \bar{u}),$$

where

$$x_{l+1} = \frac{\partial \psi_l}{\partial x_l}(\breve{x}_l, \breve{u}_l) x_l + \frac{\partial \psi_l}{\partial u_l}(\breve{x}_l, \breve{u}_l) u_l,$$

and

$$I(\bar{x}, \bar{u}) = \{k : k = 0, 1, \ldots, N-1 \text{ such that } \bar{g}_k = 0\}.$$

We now establish the second-order necessary and sufficient conditions for the problem (21)–(24) without requiring all objective functions to be of class C^2.

Theorem 4.2. *Suppose that* $(\bar{x}, \bar{u}) = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$ *is an admissible point for the problem* (21)–(24) *and assumptions* (H_1)–(H_3) *are satisfied. If* $(\bar{x}, \bar{u})$ *is a local minimum for the problem* (21)–(24)*, then the following conditions are fulfilled:*

(i) *For all* $z = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in Z$*,* $x_0 = 0$*,* $x_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k} x_k + \frac{\partial \bar{\varphi}_k}{\partial u_k} u_k$*, and* $\frac{\partial \bar{g}_k}{\partial x_k} x_k + \frac{\partial \bar{g}_k}{\partial u_k} u_k \in T(D_k; \bar{g}_k)$ $(\forall\ k = 0, 1, \ldots, N-1)$*, one has*

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} u_k \geq 0;$$

(ii) *For all* $z = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in Z$*,* $x_0 = 0$*,* $x_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k} x_k + \frac{\partial \bar{\varphi}_k}{\partial u_k} u_k$*,* $\frac{\partial \bar{g}_k}{\partial x_k} x_k + \frac{\partial \bar{g}_k}{\partial u_k} u_k \in T(D_k; \bar{g}_k)$ $(\forall\ k = 0, 1, \ldots, N-1)$*, and* $\sum\limits_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} x_k + \sum\limits_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} u_k = 0$*, one has*

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} \tilde{x}_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} \tilde{u}_k \geq 0,$$

for all $\tilde{z} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_N, \tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{N-1}) \in Z$*,* $\sum\limits_{k=1}^{N} x_k \tilde{x}_k + \sum\limits_{k=1}^{N-1} u_k \tilde{u}_k = 0$*,* $\tilde{x}_0 = 0$*,* $\tilde{x}_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k} \tilde{x}_k + \frac{\partial \bar{\varphi}_k}{\partial u_k} \tilde{u}_k$*,* $\frac{\partial \bar{g}_k}{\partial x_k} \tilde{x}_k + \frac{\partial \bar{g}_k}{\partial u_k} \tilde{u}_k \in T''\Big(D_k; \bar{g}_k, \frac{\partial \bar{g}_k}{\partial x_k} x_k + \frac{\partial \bar{g}_k}{\partial u_k} u_k\Big)$ $(\forall\ k = 0, 1, \ldots, N-1)$*, and there exist* $z_0^* = (x_0^*, u_0^*) \in \partial_S^2 h_0(\bar{x}_0, \bar{u}_0)(x_0, u_0)$ *such that*

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} \hat{x}_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} \hat{u}_k + \langle (x_0^*, u_0^*), (x_0, u_0) \rangle + \sum_{k=1}^{N-1} (x_k, u_k)^T \nabla^2 \bar{h}_k (x_k, u_k) + x_N^T \nabla^2 \bar{h}_N x_N \geq 0,$$

for all $\hat{z} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N, \hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{N-1}) \in Z$, $\sum_{k=1}^{N} x_k \hat{x}_k + \sum_{k=1}^{N-1} u_k \hat{u}_k = 0$,

$$\begin{cases} \hat{x}_0 = 0 \\ \hat{x}_{k+1} - \frac{\partial \bar{\varphi}_k}{\partial x_k}\hat{x}_k - \frac{\partial \bar{\varphi}_k}{\partial u_k}\hat{u}_k - x_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial x_k^2} x_k - x_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial x_k \partial u_k} u_k - u_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial u_k \partial x_k} x_k - u_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial u_k^2} u_k = 0 \\ \qquad (\forall\ k = 0, 1, \ldots, N-1), \end{cases}$$

$$\frac{\partial \bar{g}_k}{\partial x_k}\hat{x}_k + \frac{\partial \bar{g}_k}{\partial u_k}\hat{u}_k \in T^2\Big(D_k; \bar{g}_k, \frac{\partial \bar{g}_k}{\partial x_k}x_k + \frac{\partial \bar{g}_k}{\partial u_k}u_k - x_k^T \frac{\partial^2 \bar{g}_k}{\partial x_k^2}x_k - x_k^T \frac{\partial^2 \bar{g}_k}{\partial x_k \partial u_k}u_k - u_k^T \frac{\partial^2 \bar{g}_k}{\partial u_k \partial x_k}x_k - u_k^T \frac{\partial^2 \bar{g}_k}{\partial u_k^2}u_k\Big)\ (\forall\ k = 0, 1, \ldots, N-1).$$

The above conditions are also sufficient for $(\bar{x}, \bar{u})$ *is a local minimum for the problem* (21)–(24) *if the following condition holds:*
(ii') *For all* $z = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in Z\backslash\{0\}$, $x_0 = 0, x_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k}x_k + \frac{\partial \bar{\varphi}_k}{\partial u_k}u_k$, $\frac{\partial \bar{g}_k}{\partial x_k}x_k + \frac{\partial \bar{g}_k}{\partial u_k}u_k \in T(D_k; \bar{g}_k)$ $(\forall\ k = 0, 1, \ldots, N-1)$, $\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k}x_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k}u_k = 0$, *one has*

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k}\tilde{x}_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k}\tilde{u}_k > 0,$$

for all $\tilde{z} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_N, \tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{N-1}) \in Z\backslash\{0\}$, $\sum_{k=1}^{N} x_k\tilde{x}_k + \sum_{k=1}^{N-1} u_k\tilde{u}_k = 0$, $\tilde{x}_0 = 0$, $\tilde{x}_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k}\tilde{x}_k + \frac{\partial \bar{\varphi}_k}{\partial u_k}\tilde{u}_k$, $\frac{\partial \bar{g}_k}{\partial x_k}\tilde{x}_k + \frac{\partial \bar{g}_k}{\partial u_k}\tilde{u}_k \in T''\Big(D_k; \bar{g}_k, \frac{\partial \bar{g}_k}{\partial x_k}x_k + \frac{\partial \bar{g}_k}{\partial u_k}u_k\Big)$ $(\forall\ k = 0, 1, \ldots, N-1)$, *and*

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k}\hat{x}_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k}\hat{u}_k + \langle (x_0^*, u_0^*), (x_0, u_0)\rangle + \sum_{k=1}^{N-1} (x_k, u_k)^T \nabla^2 \bar{h}_k (x_k, u_k) + x_N^T \nabla^2 \bar{h}_N x_N > 0,$$

for all $z_0^* = (x_0^*, u_0^*) \in \partial_S^2 h_0(\bar{x}_0, \bar{u}_0)(x_0, u_0)$,
$\hat{z} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N, \hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{N-1}) \in Z$, $\sum_{k=1}^{N} x_k\hat{x}_k + \sum_{k=1}^{N-1} u_k\hat{u}_k = 0$,

$$\begin{cases} \hat{x}_0 = 0 \\ \hat{x}_{k+1} - \frac{\partial \bar{\varphi}_k}{\partial x_k}\hat{x}_k - \frac{\partial \bar{\varphi}_k}{\partial u_k}\hat{u}_k - x_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial x_k^2} x_k - x_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial x_k \partial u_k} u_k - u_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial u_k \partial x_k} x_k - u_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial u_k^2} u_k = 0 \\ \qquad (\forall\ k = 0, 1, \ldots, N-1), \end{cases}$$

$$\frac{\partial \bar{g}_k}{\partial x_k}\hat{x}_k + \frac{\partial \bar{g}_k}{\partial u_k}\hat{u}_k \in T^2\Big(D_k; \bar{g}_k, \frac{\partial \bar{g}_k}{\partial x_k}x_k + \frac{\partial \bar{g}_k}{\partial u_k}u_k - x_k^T \frac{\partial^2 \bar{g}_k}{\partial x_k^2}x_k - x_k^T \frac{\partial^2 \bar{g}_k}{\partial x_k \partial u_k}u_k - u_k^T \frac{\partial^2 \bar{g}_k}{\partial u_k \partial x_k}x_k - u_k^T \frac{\partial^2 \bar{g}_k}{\partial u_k^2}u_k\Big)\ (\forall\ k=0, 1, \ldots, N-1).$$

Remark 4.3. To establish sufficient conditions, the authors in [36] required the functions $\frac{\partial \bar{g}_k}{\partial u_k} : U_k \to \mathbf{R}$ to be surjective, even when all objective functions were of class C^2. In contrast, we achieve sufficient conditions using only assumption (H_3). This assumption is weaker than the surjectivity requirement of [36], and notably, our result holds even if one of objective functions lacks C^2 smoothness.

If at least one objective function is not of class C^2, the authors [39] derived only directional first-order necessary optimality conditions for the discrete optimal control problem (21)–(24). In contrast, Theorem 4.2 establishes second-order necessary and sufficient optimality conditions for the same problem under this assumption. Thus, our results significantly strengthen those in [39] by providing higher-order conditions that are also sufficient.

We now rewrite the problem (21)–(24) in the form

$$\begin{cases} \text{Minimize} \quad f(z) \\ \text{subject to} \quad z \in A \cap F^{-1}(D), \end{cases}$$

where F and A are defined by (25) and (27), respectively. The next step is to apply Theorem 3.1 and Theorem 3.4 to prove Theorem 4.2.

Proof of Theorem 4.2. Assume that $\bar{z} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$ is an admissible point for the problem (21)–(24) and assumptions (H_1)–(H_3) are satisfied. From [37, Lemma 5.1], we get $T\big(D; F(\bar{z})\big) = D\big(F(\bar{z})\big)$. By [38, Lemma 7], we have that $\nabla H(\bar{z})$ is surjective and the regularity condition (1) is satisfied. If $\bar{z} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$ is a local minimum for the problem (21)–(24), then $\bar{z}$ is a locally optimal solution of problem (26). Since

$$f(z) = f(x, u) = \sum_{k=0}^{N-1} h_k(x_k, u_k) + h_N(x_N),$$

we have

$$\begin{aligned} &\nabla f(\bar{z}) = \nabla f(\bar{x}, \bar{u}) \\ &= \Big(\frac{\partial h_0}{\partial x_0}(\bar{x}_0, \bar{u}_0), \frac{\partial h_1}{\partial x_1}(\bar{x}_1, \bar{u}_1), \ldots, \frac{\partial h_{N-1}}{\partial x_{N-1}}(\bar{x}_{N-1}, \bar{u}_{N-1}), \frac{\partial h_N}{\partial x_N}(\bar{x}_N), \\ &\qquad \frac{\partial h_0}{\partial u_0}(\bar{x}_0, \bar{u}_0), \frac{\partial h_1}{\partial u_1}(\bar{x}_1, \bar{u}_1), \ldots, \frac{\partial h_{N-1}}{\partial u_{N-1}}(\bar{x}_{N-1}, \bar{u}_{N-1}) \Big). \end{aligned}$$

So, for each $z = (x, u) = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in Z$ we get

$$\langle \nabla f(\bar{z}), z \rangle = \sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} u_k, \tag{28}$$

$$\nabla H(\bar{z})z = \Big(x_0,\ x_1 - \frac{\partial \bar{\varphi}_0}{\partial x_0}x_0 - \frac{\partial \bar{\varphi}_0}{\partial u_0}u_0,\ x_2 - \frac{\partial \bar{\varphi}_1}{\partial x_1}x_1 - \frac{\partial \bar{\varphi}_1}{\partial u_1}u_1, \ldots, x_N - \frac{\partial \bar{\varphi}_{N-1}}{\partial x_{N-1}}x_{N-1} - \frac{\partial \bar{\varphi}_{N-1}}{\partial u_{N-1}}u_{N-1}\Big), \tag{29}$$

and

$$\nabla F(\bar{z})z = \Big(\frac{\partial \bar{g}_0}{\partial x_0}x_0 + \frac{\partial \bar{g}_0}{\partial u_0}u_0, \frac{\partial \bar{g}_1}{\partial x_1}x_1 + \frac{\partial \bar{g}_1}{\partial u_1}u_1, \ldots, \frac{\partial \bar{g}_{N-1}}{\partial x_{N-1}}x_{N-1} + \frac{\partial \bar{g}_{N-1}}{\partial u_{N-1}}u_{N-1}\Big). \tag{30}$$

We also get

$$z^T\nabla^2 H(\bar{z})z = -\Big(0, x_0^T\frac{\partial^2 \bar{\varphi}_0}{\partial x_0^2}x_0 + x_0^T\frac{\partial^2 \bar{\varphi}_0}{\partial x_0\partial u_0}u_0 + u_0^T\frac{\partial^2 \bar{\varphi}_0}{\partial u_0\partial x_0}x_0 + u_0^T\frac{\partial^2 \bar{\varphi}_0}{\partial u_0^2}u_0, \ldots, x_{N-1}^T\frac{\partial^2 \bar{\varphi}_{N-1}}{\partial x_{N-1}^2}x_{N-1} + x_{N-1}^T\frac{\partial^2 \bar{\varphi}_{N-1}}{\partial x_{N-1}\partial u_{N-1}}u_{N-1} + u_{N-1}^T\frac{\partial^2 \bar{\varphi}_{N-1}}{\partial u_{N-1}\partial x_{N-1}}x_{N-1} + u_{N-1}^T\frac{\partial^2 \bar{\varphi}_{N-1}}{\partial u_{N-1}^2}u_{N-1}\Big) \tag{31}$$

and

$$z^T\nabla^2 F(\bar{z})z = \Big(x_0^T\frac{\partial^2 \bar{g}_0}{\partial x_0^2}x_0 + x_0^T\frac{\partial^2 \bar{g}_0}{\partial x_0\partial u_0}u_0 + u_0^T\frac{\partial^2 \bar{g}_0}{\partial u_0\partial x_0}x_0 + u_0^T\frac{\partial^2 \bar{g}_0}{\partial u_0^2}u_0, x_1^T\frac{\partial^2 \bar{g}_1}{\partial x_1^2}x_1 + x_1^T\frac{\partial^2 \bar{g}_1}{\partial x_1\partial u_1}u_1 + u_1^T\frac{\partial^2 \bar{g}_1}{\partial u_1\partial x_1}x_1 + u_1^T\frac{\partial^2 \bar{g}_1}{\partial u_1^2}u_1, \ldots, x_{N-1}^T\frac{\partial^2 \bar{g}_{N-1}}{\partial x_{N-1}^2}x_{N-1} + x_{N-1}^T\frac{\partial^2 \bar{g}_{N-1}}{\partial x_{N-1}\partial u_{N-1}}u_{N-1} + u_{N-1}^T\frac{\partial^2 \bar{g}_{N-1}}{\partial u_{N-1}\partial x_{N-1}}x_{N-1} + u_{N-1}^T\frac{\partial^2 \bar{g}_{N-1}}{\partial u_{N-1}^2}u_{N-1}\Big). \tag{32}$$

Since Lemma 2.8 for each $z^* \in \partial_S^2 f(\bar{z})(z)$, there exists $z_0^* = (x_0^*, u_0^*) \in \partial_S^2 h_0(\bar{x}_0, \bar{u}_0)(x_0, u_0)$ such that

$$\langle z^*, z\rangle = \langle (x_0^*, u_0^*), (x_0, u_0)\rangle + \sum_{k=1}^{N-1}(x_k, u_k)^T\nabla^2\bar{h}_k(x_k, u_k) + x_N^T\nabla^2\bar{h}_N x_N. \tag{33}$$

By Theorem 3.1, the following assertions hold:
(a') $\langle \nabla f(\bar{z}), z\rangle \geq 0$ for all $z \in Z : \nabla H(\bar{z})z = 0$ and $\nabla F(\bar{z})z \in T(D; F(\bar{z}))$;
(b') For any $z \in Z$, $\nabla H(\bar{z})z = 0$, $\nabla F(\bar{z})z \in T(D; F(\bar{z}))$, and $\langle \nabla f(\bar{z}), z\rangle = 0$, one has

$$\langle \nabla f(\bar{z}), \tilde{z}\rangle \geq 0\ \forall \tilde{z} \in z^\perp,\ \nabla H(\bar{z})\tilde{z} = 0,\ \nabla F(\bar{z})\tilde{z} \in T^{''}\big(D; F(\bar{z}), \nabla F(\bar{z})z\big)$$

and there is $z^* \in \partial_S^2 f(\bar{z})(z)$ such that

$$\langle \nabla f(\bar{z}), \hat{z}\rangle + \langle z^*, z\rangle \geq 0, \forall \hat{z} \in z^\perp, \text{ and } \nabla H(\bar{z})\hat{z} + z^T\nabla^2 H(\bar{z})z = 0, \nabla F(\bar{z})\hat{z} \in T^2\big(D; F(\bar{z}), \nabla F(\bar{z})z - z^T\nabla^2 F(\bar{z})z\big).$$

Since (28)–(30) and Lemma 2.7, the assertion (a') implies that for all

$$z = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in Z,$$

$$\begin{cases} x_0 = 0 \\ x_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k} x_k + \frac{\partial \bar{\varphi}_k}{\partial u_k} u_k, \ \forall \ k = 0, 1, \ldots, N-1 \end{cases}$$

and

$$\frac{\partial \bar{g}_k}{\partial x_k} x_k + \frac{\partial \bar{g}_k}{\partial u_k} u_k \in T(D_k; \bar{g}_k), \ \forall \ k = 0, 1, \ldots, N-1,$$

one has

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} u_k \geq 0,$$

this is the assertion (i) of Theorem 4.2.

Since (31)–(33) and Lemma 2.7, Lemma 2.8, the assertion (b') implies that for all $z = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in Z$,

$$\begin{cases} x_0 = 0 \\ x_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k} x_k + \frac{\partial \bar{\varphi}_k}{\partial u_k} u_k, \ \forall \ k = 0, 1, \ldots, N-1, \end{cases}$$

$$\frac{\partial \bar{g}_k}{\partial x_k} x_k + \frac{\partial \bar{g}_k}{\partial u_k} u_k \in T(D_k; \bar{g}_k), \ \forall \ k = 0, 1, \ldots, N-1$$

and

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} u_k = 0,$$

one has

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} \tilde{x}_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} \tilde{u}_k \geq 0,$$

for all $\tilde{z} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_N, \tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{N-1}) \in Z$,

$$\sum_{k=1}^{N} x_k \tilde{x}_k + \sum_{k=1}^{N-1} u_k \tilde{u}_k = 0,$$

$$\begin{cases} \tilde{x}_0 = 0 \\ \tilde{x}_{k+1} = \frac{\partial \bar{\varphi}_k}{\partial x_k} \tilde{x}_k + \frac{\partial \bar{\varphi}_k}{\partial u_k} \tilde{u}_k, \ \forall \ k = 0, 1, \ldots, N-1, \end{cases}$$

$$\frac{\partial \bar{g}_k}{\partial x_k} \tilde{x}_k + \frac{\partial \bar{g}_k}{\partial u_k} \tilde{u}_k \in T^{''}\Big(D_k; \bar{g}_k, \frac{\partial \bar{g}_k}{\partial x_k} x_k + \frac{\partial \bar{g}_k}{\partial u_k} u_k\Big), \ \forall \ k = 0, 1, \ldots, N-1,$$

and there exist $z_0^* = (x_0^*, u_0^*) \in \partial_S^2 h_0(\bar{x}_0, \bar{u}_0)(x_0, u_0)$ such that

$$\sum_{k=0}^{N} \frac{\partial \bar{h}_k}{\partial x_k} \hat{x}_k + \sum_{k=0}^{N-1} \frac{\partial \bar{h}_k}{\partial u_k} \hat{u}_k + \langle (x_0^*, u_0^*), (x_0, u_0) \rangle + \sum_{k=1}^{N-1} (x_k, u_k)^T \nabla^2 \bar{h}_k (x_k, u_k)$$
$$+ x_N^T \nabla^2 \bar{h}_N x_N \geq 0,$$

for all $\hat{z} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N, \hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{N-1}) \in Z$,

$$\sum_{k=1}^{N} x_k \hat{x}_k + \sum_{k=1}^{N-1} u_k \hat{u}_k = 0,$$

$$\begin{cases} \hat{x}_0 = 0 \\ \hat{x}_{k+1} - \frac{\partial \bar{\varphi}_k}{\partial x_k}\hat{x}_k - \frac{\partial \bar{\varphi}_k}{\partial u_k}\hat{u}_k - x_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial x_k^2} x_k - x_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial x_k \partial u_k} u_k - u_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial u_k \partial x_k} x_k - u_k^T \frac{\partial^2 \bar{\varphi}_k}{\partial u_k^2} u_k = 0, \\ \qquad \forall\ k = 0, 1, \ldots, N-1, \end{cases}$$

$$\frac{\partial \bar{g}_k}{\partial x_k}\hat{x}_k + \frac{\partial \bar{g}_k}{\partial u_k}\hat{u}_k \in T^2\Big(D_k; \bar{g}_k, \frac{\partial \bar{g}_k}{\partial x_k}x_k + \frac{\partial \bar{g}_k}{\partial u_k}u_k - x_k^T \frac{\partial^2 \bar{g}_k}{\partial x_k^2} x_k - x_k^T \frac{\partial^2 \bar{g}_k}{\partial x_k \partial u_k} u_k - u_k^T \frac{\partial^2 \bar{g}_k}{\partial u_k \partial x_k} x_k - u_k^T \frac{\partial^2 \bar{g}_k}{\partial u_k^2} u_k\Big),\ \forall\ k = 0, 1, \ldots, N-1,$$

this is the assertion (ii) of Theorem 4.2. By Theorem 3.4, we can prove the sufficient optimality conditions for the problem (21)–(24). The proof of Theorem 4.2 is complete. □

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Declarations

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