Evolution Equations and Control Theory
Vol. 14, No. 5, October 2025, pp. 1167-1189
doi:10.3934/eect.2025030

SENSITIVITY ANALYSIS FOR EQUILIBRIUM PROBLEMS AND APPLICATIONS TO OPTIMAL CONTROL PROBLEMS

Lam Quoc Anh[1], Nguyen Huu Danh[2], Pham Thanh Duoc[3]
and Thanatporn Grace*[4]

[1]Department of Mathematics, Teacher College, Cantho University, Vietnam

[2]Department of Mathematics, Taydo University, Cantho, Vietnam

[3]Faculty of Information Technology, Cantho University of Technology, Cantho, Vietnam

[4]School of Science, Walailak University, Tha Sala District,
Nakhon Si Thammarat 80160, Thailand

(Communicated by Ewa Bednarczuk)

Abstract. In this paper, we consider parametric set-valued equilibrium problems in normed spaces. By employing the direct approach based on generalized convexity and monotonicity assumptions, we establish Hölder/Lipschitz conditions for both exact and approximate efficient solutions of the reference problems. We demonstrate that with this approach, the commonly required additional conditions of indirect methods, such as scalarization methods, as seen in existing works, can be avoided. Utilizing the proposed approach and techniques, we also derive Lipschitz conditions for two optimal control models in biology and economics: one describing the interaction between a predator and its prey, and another addressing the balance between holding cash and investing.

1. Introduction.

The equilibrium problem holds significant importance in various fields and serves as a unifying framework for many important problems in optimization theory, including variational inequalities, game theory, mathematical economics, optimization problems, and fixed point theory [13,16,33,39,46]. Although most authors claim that the term "*equilibrium problem*" was first introduced by Blum and Oettli [21] in 1994, it actually appeared two years earlier in a paper by Muu and Oettli [38]. However, it is widely agreed that the problem truly began its most significant and sustained development phase with the paper by Blum and Oettli [21], which is likely why many regard it as the starting point for the field. Since then, the equilibrium problem model has been significantly extended to address various practical demands [2, 3, 22]. Notable works in this area include the vector equilibrium problem [8, 23, 30], the vector equilibrium problem with a set-valued objective map [5, 17], the variational inclusion problems [11, 29], and the variational relation problems [19, 36].

2020 *Mathematics Subject Classification.* 49K40, 90C31, 91B50, 93C15.
Key words and phrases. Equilibrium problem, optimal control problem, Hölder/Lipschitz continuity, convexity, monotonicity.
*Corresponding author: Thanatporn Grace.

One of the research topics for optimization models that has attracted significant attention from researchers is the Hölder/Lipschitz property of solutions, and naturally, the equilibrium problem is no exception. To study the properties of solutions of this type for the equilibrium problem, mathematicians have so far employed two main approaches. The first is the direct approach, in which researchers establish the Hölder/Lipschitz property by imposing conditions related to strong monotonicity [10, 20] or strong convexity [6, 42] of the objective maps with respect to exact solutions. In the case of approximate solutions, these strong conditions are weakened to monotonicity or convexity conditions [12, 30]. The second is the indirect approach, which follows a general research procedure that can be summarized as follows. First, researchers construct scalarization functions compatible with the types of solutions under consideration. Next, they examine the fundamental properties of these scalarization functions, such as convexity, monotonicity, and continuity. Finally, the properties of the scalarization functions are applied to represent the solutions of the vector problem and to establish Hölder/Lipschitz conditions for these solutions. The advantage of this second approach lies in its clear and structured process, making it relatively easy to apply. However, its drawback is the additional cost incurred in transforming (or representing) the solutions of the original problem into scalar problems. This transformation requires supplementary conditions, such as convexlikeness conditions for the linear scalarization method [30, 40] or cone solidness condition for the nonlinear scalarization method [14, 15]. Returning to the equilibrium problems with set-valued objective maps, often referred to simply as the set-valued equilibrium problem, due to technical difficulties, research on Hölder/Lipschitz conditions for such problems remains limited. Most studies focus on weak solutions or ideal solutions, as the scalarization method is particularly suitable for these types of solutions [15, 20, 34]. For efficient solutions, however, the problem is much harder to handle. The first approach often involves evaluating whether the sum of two vectors belonging to the complement of a convex cone still does not belong to the cone. Meanwhile, for the scalarization method, a suitable scalarization function has yet to be found, further complicating the analysis for efficient solutions.

From the review of research on Hölder/Lipschitz conditions for equilibrium problems, as mentioned above, we identify the main objective of this study as investigating the Hölder/Lipschitz properties of both exact and approximate solutions to set-valued equilibrium problems using the first approach. This approach aims to eliminate the supplementary conditions of the second approach and relax the strong conditions so that the results obtained can be applied to optimization models in practice. Specifically, as follows: In Section 2, we recall concepts and results related to the monotonicity, convexity, and Hölder/Lipschitz properties of single-valued and set-valued maps, which will be used in the subsequent sections. Section 3, the main part of this study, presents the results on the Hölder/Lipschitz conditions for both exact and approximate efficient solutions of set-valued equilibrium problems. Section 4 focuses on the study of Hölder/Lipschitz properties of optimal control problems in biology and economics, including a problem describing the interaction between a predator and its prey, and another addressing the balance between holding cash and investing. Finally, Section 5 provides conclusions and discussions related to the results achieved in this study.

2. **Preliminaries.** Let $\mathbb{X}$, $\mathbb{Y}$ be normed spaces, C be a pointed, convex cone in $\mathbb{Y}$, and $\mathbb{B}_{\mathbb{Y}}$ be the closed unit ball in $\mathbb{Y}$. We denote the set of all nonnegative real numbers by $\mathbb{R}_+$.

We first recall the Hölder/Lipschitz continuity concepts of a set-valued map.

Definition 2.1. (see [31]) Let Ω be a nonempty subset of $\mathbb{X}$. A set-valued map $G : \Omega \rightrightarrows \mathbb{Y}$ is said to be

(a) *locally Hölder continuous* at $x_0 \in \Omega$ if there exist positive real numbers ℓ, α and a neighborhood U of x_0 such that

$$G(x_1) \subseteq G(x_2) + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_{\mathbb{Y}} \quad \text{for all } x_1, x_2 \in U \cap \Omega,$$

or equivalently

$$H(G(x_1), G(x_2)) \leq \ell\|x_1 - x_2\|^\alpha,$$

where H is the Hausdorff distance between two sets.

(b) *upper locally Hölder continuous* at $x_0 \in \Omega$ if there exist positive real numbers ℓ, α and a neighborhood U of x_0 such that

$$G(x_1) \subseteq G(x_2) + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_{\mathbb{Y}} - C \quad \text{for all } x_1, x_2 \in U \cap \Omega;$$

(c) *locally Hölder continuous* on Ω if G is locally Hölder continuous at every point $x \in \Omega$;

(d) *upper locally Hölder continuous* on Ω if G is upper locally Hölder continuous at every point $x \in \Omega$;

(e) *globally Hölder continuous* on Ω if there exist positive real numbers ℓ, α such that

$$G(x_1) \subseteq G(x_2) + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_{\mathbb{Y}} \quad \text{for all } x_1, x_2 \in \Omega;$$

(f) *upper globally Hölder continuous* on Ω if there exist positive real numbers ℓ, α such that

$$G(x_1) \subseteq G(x_2) + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_{\mathbb{Y}} - C \quad \text{for all } x_1, x_2 \in \Omega.$$

If $\alpha = 1$, then the Hölder continuity reduces to the Lipschitz continuity.

Remark 2.2. Obviously, if G is (upper) globally Hölder continuous on Ω, then it is also (upper) locally Hölder continuous on Ω. Moreover, if G is globally/locally Hölder continuous on Ω, then G is also upper globally/locally Hölder continuous on Ω.

The following examples indicate that the reverse of the aforementioned statements generally does not occur.

Example 2.3. (Upper globally Hölder continuity but not globally Hölder continuitys)

Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ and $\Omega = C = \mathbb{R}_+$. Consider the set-valued map $G : \Omega \rightrightarrows \mathbb{R}$ by

$$G(x) = \begin{cases} \{0\}, & x = 0, \\ \left[-\frac{1}{x}, x\right], & x \neq 0. \end{cases}$$

It is easy to see that G is upper globally Hölder continuous on Ω with $\ell = 1$ and $\alpha = 1$, but not globally Hölder continuous on Ω.

Example 2.4. Let $\mathbb{X} = \mathbb{Y} = \Omega = \mathbb{R}$ and $C = \mathbb{R}_+$. The set-valued map $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is defined as follows

$$G(x) = \begin{cases} [x-1, x] & \text{if } x \neq 0, \\ -\mathbb{R}_+ & \text{if } x = 0. \end{cases}$$

It is easy to see that G is upper locally Lipschitz continuous at $0 \in \mathbb{R}$, but G is not locally Lipschitz continuous at $0 \in \mathbb{R}$.

Lemma 2.5. *Let Ω be a nonempty subset of $\mathbb{X}$ and $Q : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued map defined by*

$$Q(x) = G(x) - C \quad \text{for all } x \in \mathbb{X}.$$

Then, the map G is upper (globally/locally) Hölder continuous on Ω if and only if Q is (globally/locally) Hölder continuous on Ω.

Proof. Since the proof is similar, we only show one statement. Let $x_0 \in \Omega$ be arbitrary.

(Sufficient condition): Since the map G is upper locally Hölder continuous at x_0, there exist positive real numbers ℓ, α and a neighborhood U of x_0 such that

$$G(x_1) \subseteq G(x_2) + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_\mathbb{Y} - C \quad \text{for all } x_1, x_2 \in U \cap \Omega.$$

Combining this with the convexity of C, we obtain

$$\begin{aligned} G(x_1) - C &\subseteq G(x_2) - C + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_\mathbb{Y} - C \\ &\subseteq G(x_2) - C + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_\mathbb{Y}, \end{aligned}$$

for all $x_1, x_2 \in U \cap \Omega$. It means that

$$Q(x_1) \subseteq Q(x_2) + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_\mathbb{Y} \quad \text{for all } x_1, x_2 \in U \cap \Omega.$$

(Necessary condition): By the same above arguments, we can find positive real numbers ℓ, α and a neighborhood U of x_0 such that

$$G(x_1) - C \subseteq G(x_2) - C + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_\mathbb{Y} \quad \text{for all } x_1, x_2 \in U \cap \Omega.$$

This together with $0 \in C$ would imply that

$$G(x_1) \subseteq G(x_2) + \ell\|x_1 - x_2\|^\alpha \mathbb{B}_\mathbb{Y} - C \quad \text{for all } x_1, x_2 \in U \cap \Omega,$$

and hence the proof is complete. □

Next, we recall some concepts of cone-convexity for a set-valued map, which are used to study stability conditions in the next section.

Definition 2.6. (see [41]) Let Ω be a nonempty convex subset of $\mathbb{X}$. A set-valued map $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be

(a) *lower C-convex* on Ω if for any $x_1, x_2 \in \Omega$ and $t \in [0, 1]$,

$$tG(x_2) + (1 - t)G(x_1) \subseteq G(tx_2 + (1 - t)x_1) + C;$$

(b) *upper C-convex* on Ω if for any $x_1, x_2 \in \Omega$ and $t \in [0, 1]$,

$$G(tx_2 + (1 - t)x_1) \subseteq tG(x_2) + (1 - t)G(x_1) - C;$$

(c) *lower C-quasiconvex* on Ω if for any convex subset A of $\mathbb{Y}$, $x_1, x_2 \in \Omega$ and $t \in]0, 1[$,

$$0 \in tG(x_2) + (1 - t)G(x_1) + A + C$$

implies

$$0 \in G((1 - t)x_1 + tx_2) + A + C;$$

(d) *upper C-quasiconvex* on Ω if for any convex subset A of $\mathbb{Y}$, $x_1, x_2 \in \Omega$ and $t \in]0, 1[$,

$$tG(x_2) + (1 - t)G(x_1) \subseteq A - C$$

implies

$$G((1 - t)x_1 + tx_2) \subseteq A - C.$$

Remark 2.7. Let $G : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued map and Ω be a nonempty convex subset of $\mathbb{X}$. In view of Definition 2.6, we obtain the statements below.

(a) If G is lower C-quasiconvex on Ω, then for any $x_1, x_2 \in \Omega$, $r \in \mathbb{R}_+$ and $t \in]0,1[$,

$$(tG(x_2) + (1-t)G(x_1)) \cap (r\mathbb{B}_{\mathbb{Y}} - C) \neq \emptyset$$

implies

$$(G((1-t)x_1 + tx_2)) \cap (r\mathbb{B}_{\mathbb{Y}} - C) \neq \emptyset;$$

(b) If G is upper C-quasiconvex on Ω, then for any $x_1, x_2 \in \Omega$, $r \in \mathbb{R}_+$ and $t \in]0,1[$,

$$tG(x_2) + (1-t)G(x_1) \subseteq r\mathbb{B}_{\mathbb{Y}} - C$$

implies

$$G((1-t)x_1 + tx_2) \subseteq r\mathbb{B}_{\mathbb{Y}} - C.$$

Lemma 2.8. (see [41]) *Let Ω be a nonempty convex subset of $\mathbb{X}$. If G is upper (lower) C-convex on Ω, then it is upper (lower) C-quasiconvex on Ω.*

We conclude this section by revisiting some concepts of generalized monotonicity for set-valued functions.

Definition 2.9. (see [9]) Let Ω be a nonempty subset of $\mathbb{X}$. A set-valued map $Q : \mathbb{X} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be

(a) *quasimonotone wrt* C on $\Omega \subseteq X$ if for all $x, y \in \Omega$,

$$[Q(x,y) \subseteq \mathbb{Y} \setminus C] \Longrightarrow [Q(y,x) \not\subseteq \mathbb{Y} \setminus C]\,;$$

(b) *Hölder strongly monotone wrt* C on $\Omega \subseteq X$ if there exist positive real numbers η, γ such that

$$Q(y,x) + Q(x,y) + \eta\|x-y\|^{\gamma}\mathbb{B}_{\mathbb{Y}} \subseteq -C \quad \text{for all } x, y \in \Omega;$$

(c) *Hölder strongly pseudomonotone wrt* C on Ω if there exist positive real numbers η, γ such that

$$[Q(x,y) \cap C \neq \emptyset] \Longrightarrow [\eta\|x-y\|^{\gamma}\mathbb{B}_{\mathbb{Y}} + Q(y,x) \subseteq -C] \quad \text{for all } x, y \in \Omega.$$

Remark 2.10. Let Ω be a nonempty subset of $\mathbb{X}$. If a map $Q : \mathbb{X} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is Hölder strongly pseudomonotone as well as quasimonotone wrt C on Ω, then there exist positive real numbers η, γ such that

$$\text{either } \eta\|x-y\|^{\gamma}\mathbb{B}_{\mathbb{Y}} + Q(x,y) \subseteq -C \text{ or } \eta\|x-y\|^{\gamma}\mathbb{B}_{\mathbb{Y}} + Q(y,x) \subseteq -C.$$

3. Global Hölder conditions for set-valued equilibrium problems.

In this section, we study the stability for parametric set-valued equilibrium problems. Let $\mathbb{X}$, $\mathbb{Y}$ be defined as in Section 2, and $\mathbb{W}$ be a normed space. Let Ω be a nonempty convex subset of $\mathbb{X}$, and Λ, Θ be nonempty subsets of $\mathbb{W}$. For set-valued maps with nonempty values $F : \Omega \times \Omega \times \Theta \rightrightarrows \mathbb{Y}$ and $K : \Lambda \rightrightarrows \Omega$, we consider the following parametric set-valued equilibrium problem.

(SEP): Find $\bar{x} \in K(\lambda)$ such that

$$F(\bar{x}, y, \mu) \cap C \neq \emptyset \quad \text{for all } y \in K(\lambda). \tag{1}$$

Definition 3.1. For each $(\lambda, \mu) \in \Lambda \times \Theta$, $\bar{x} \in K(\lambda)$ is called

(a) *an efficient solution* to (SEP) if (1) holds true;

(b) *an ideal solution* to (SEP) if

$$F(\bar{x}, y, \mu) \subseteq C \quad \text{for all } y \in K(\lambda);$$

(c) *a weak solution* to (SEP) if

$$F(\bar{x}, y, \mu) \cap (\mathbb{Y} \setminus -\operatorname{int} C) \neq \emptyset \quad \text{for all } y \in K(\lambda).$$

We denote the efficient solution (ideal solution, weak solution, respectively) set to (SEP) by $\mathrm{Sol_E}(\mathrm{SEP})$ ($\mathrm{Sol_I}(\mathrm{SEP}), \mathrm{Sol_W}(\mathrm{SEP})$, respectively). Then, we have the following inclusions:

$$\mathrm{Sol_I}(\mathrm{SEP}) \subseteq \mathrm{Sol_E}(\mathrm{SEP}) \subseteq \mathrm{Sol_W}(\mathrm{SEP}).$$

Since the solvability of equilibrium problems has been extensively discussed [4, 27, 32], we assume throughout this work that the problem (SEP) is solvable at every point (λ, μ). Until now, there are many studies devoted to sufficient conditions for the Hölder/Lipschitz continuity of the ideal solution map $\mathrm{Sol_I}(\mathrm{SEP})$ [30, 42] as well as the weak solution map $\mathrm{Sol_W}(\mathrm{SEP})$ [9, 10, 34]. Due to technical challenges in studying stability conditions for the efficient solution map $\mathrm{Sol_E}(\mathrm{SEP})$, to the best of our knowledge, only papers [7, 17] have examined the Hausdorff continuity of this efficient solution map. Moreover, as far as we know, there has been no research on the Hölder/Lipschitz continuity of $\mathrm{Sol_E}(\mathrm{SEP})$.

Based on existing works on the Hölder continuity of solution maps for equilibrium problems, we have identified two main approaches. The first approach examines Hölder continuity through conditions related to the strong monotonicity of the objective map [10,20], while the second investigates this property through conditions related to convexity and strong convexity [30, 42]. Motivated by these observations, we will study the Hölder continuity of the efficient solution map $\mathrm{Sol_E}(\mathrm{SEP})$ for set-valued equilibrium problems using both approaches.

3.1. **Monotone set-valued equilibrium problems.** In this subsection, we employ conditions related to the monotonicity of a set-valued map to discuss Hölder conditions for the efficient solution map of (SEP). We first study the Hölder continuity conditions for $\mathrm{Sol_E}(\mathrm{SEP})$ under perturbations in the objective map.

Theorem 3.2. *Let $\lambda \in \Lambda$ be given. Assume that*

(i) *F is upper globally Hölder continuous in the third argument on Θ;*

(ii) *for each $\mu \in \Theta$, $F(\cdot, \cdot, \mu)$ is Hölder strongly pseudomonotone wrt C on $K(\lambda)$.*

Then, $\mathrm{Sol_E}(\mathrm{SEP})(\lambda, \cdot)$ is globally Hölder continuous on Θ.

Proof. Let $\mu_1, \mu_2 \in \Theta$ be arbitrary. Taking arbitrary $x_1 \in \mathrm{Sol_E}(\mathrm{SEP})(\lambda, \mu_1)$ and $x_2 \in \mathrm{Sol_E}(\mathrm{SEP})(\lambda, \mu_2)$, for all $y \in \Omega$, we have then

$$F(x_1, y, \mu_1) \cap C \neq \emptyset \quad \text{and} \quad F(x_2, y, \mu_2) \cap C \neq \emptyset.$$

Equivalently,

$$0 \in F(x_1, y, \mu_1) - C \quad \text{and} \quad 0 \in F(x_2, y, \mu_2) - C. \tag{2}$$

It follows from (i) that there exist positive real numbers ℓ, α such that

$$F(x_2, x_1, \mu_2) \subseteq F(x_2, x_1, \mu_1) + \ell \|\mu_1 - \mu_2\|^{\alpha} \mathbb{B}_{\mathbb{Y}} - C. \tag{3}$$

By the Hölder strong pseudomonotonicity of F wrt C on Ω and $0 \in F(x_1, x_2, \mu_1) - C$, there exist positive real numbers η, γ such that

$$\eta \|x_1 - x_2\|^{\gamma} \mathbb{B}_{\mathbb{Y}} + F(x_2, x_1, \mu_1) \subseteq -C.$$

Combining this with (2) and (3), we obtain

$$\eta \|x_1 - x_2\|^{\gamma} \mathbb{B}_{\mathbb{Y}} \subseteq \eta \|x_1 - x_2\|^{\gamma} \mathbb{B}_{\mathbb{Y}} + F(x_2, x_1, \mu_2) - C$$

$$\begin{aligned}&\subseteq \eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}} + F(x_2,x_1,\mu_1) + \ell\|\mu_1-\mu_2\|^\alpha\mathbb{B}_{\mathbb{Y}} - C\\ &\subseteq \ell\|\mu_1-\mu_2\|^\alpha\mathbb{B}_{\mathbb{Y}} - C.\end{aligned}$$

Then, $\eta\|x_1-x_2\|^\gamma \le \ell\|\mu_1-\mu_2\|^\alpha$, and so

$$\|x_1-x_2\| \le \sqrt[\gamma]{\frac{\ell}{\eta}}\,\|\mu_1-\mu_2\|^{\alpha/\gamma}.$$

Therefore, the proof is complete. ☐

We next examine the conditions for Hölder continuity of $\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})$ with a parametric constraint map.

Theorem 3.3. *Let $\mu\in\Theta$ be given. Assume that*

(i) *K is globally Hölder continuous on Λ;*
(ii) *F is upper globally Hölder continuous in the second argument on $K(\Lambda)$;*
(iii) *$F(\cdot,\cdot,\mu)$ is Hölder strongly pseudomonotone as well as quasimonotone wrt C on $K(\Lambda)$.*

Then, $\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})(\cdot,\mu)$ *is globally Hölder continuous on Λ.*

Proof. Let λ_1,λ_2 be arbitrary elements in Λ. For $x_1=\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})(\lambda_1,\mu)$ and $x_2=\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})(\lambda_2,\mu)$, we will provide an estimation for $\|x_1-x_2\|$. We first consider the case where x_1 and x_2 are distinct. Since x_1,x_2 are efficient solutions of (SEP), for all $y_1\in K(\lambda_1)$ and $y_2\in K(\lambda_2)$,

$$0\in F(x_1,y_1,\mu)-C \quad\text{and}\quad 0\in F(x_2,y_2,\mu)-C. \tag{4}$$

The global Hölder continuity of K leads to the existence of positive real numbers $\hbar,\beta$ and vectors $z_1\in K(\lambda_1)$, $z_2\in K(\lambda_2)$ such that

$$\max\{\|x_1-z_2\|,\|x_2-z_1\|\}\le \hbar\|\lambda_1-\lambda_2\|^\beta. \tag{5}$$

In view of (ii), we also have positive real numbers ℓ,α such that

$$F(x_1,z_1,\mu)\subseteq F(x_1,x_2,\mu)+\ell\|x_2-z_1\|^\alpha\mathbb{B}_{\mathbb{Y}}-C. \tag{6}$$

Thanks to (iii) and Remark 2.10, there exist positive real numbers η,γ such that

either $F(x_1,x_2,\mu)+\eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}\subseteq -C$ or $F(x_2,x_1)+\eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}\subseteq -C$.

We consider two cases as follows.

Case 1. $F(x_1,x_2,\mu)+\eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}\subseteq -C$: In view of (4), (5), and (6), we have

$$\begin{aligned}\eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}} &\subseteq \eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}+F(x_1,z_1,\mu)-C\\ &\subseteq \eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}+F(x_1,x_2,\mu)+\ell\|x_2-z_1\|^\alpha\mathbb{B}_{\mathbb{Y}}-C\\ &\subseteq \ell\hbar^\alpha\|\lambda_1-\lambda_2\|^{\alpha\beta}\mathbb{B}_{\mathbb{Y}}-C.\end{aligned}$$

Case 2. $F(x_2,x_1)+\eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}\subseteq -C$: It follows from (4), (5), and (6) that

$$\begin{aligned}\eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}} &\subseteq \eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}+F(x_2,z_2,\mu)-C\\ &\subseteq \eta\|x_1-x_2\|^\gamma\mathbb{B}_{\mathbb{Y}}+F(x_2,x_1,\mu)+\ell\|z_2-x_1\|^\alpha\mathbb{B}_{\mathbb{Y}}-C\\ &\subseteq \ell\hbar^\alpha\|\lambda_1-\lambda_2\|^{\alpha\beta}\mathbb{B}_{\mathbb{Y}}-C.\end{aligned}$$

Therefore,

$$\|x_1-x_2\|\le\sqrt[\gamma]{\frac{\ell\hbar^\alpha}{\eta}}\,\|\lambda_1-\lambda_2\|^{\alpha\beta/\gamma}. \tag{7}$$

If $x_1 = x_2$, then of course (7) is also true, and so $\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})(\cdot, \mu)$ is globally Hölder continuous on Λ. □

We will conclude this subsection with a result on the Hölder continuity of the efficient solution map $\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})$ for equilibrium problems under perturbations to both the constraint and objective maps.

Theorem 3.4. *Assume that*

(i) K *is globally Hölder continuous on* Λ;
(ii) F *is upper globally Hölder continuous in the third argument on* Θ;
(iii) F *is upper globally Hölder continuous in the second argument on* $K(\Lambda)$;
(iv) $F(\cdot, \cdot, \mu)$ *is Hölder strongly pseudomonotone as well as quasimonotone wrt* C *on* $K(\Lambda)$.

Then, $\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})(\cdot, \cdot)$ *is globally Hölder continuous on* Λ.

Remark 3.5. As far as we know, the Hölder continuity of the efficient solution map $\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})$ for set-valued equilibrium problems has not been previously investigated. Therefore, to compare and clarify the novelty of our results with existing works, we consider the special case where F is a single-valued map, and then the problem (SEP) has become (VEP) considered in [6]. For this special case, the main distinction in our findings, Theorems 3.2–3.4, in this subsection is that we have established Hölder conditions for the problem under strongly pseudomonotone assumptions instead of the strongly convexity as in Theorems 3.1 and 3.6 of [6], and furthermore the monotonicity of the objective map has been reduced to quasimonotonicity. More especially, when the cone C generates a total order relation in $\mathbb{Y}$.

Example 3.6. Let $\mathbb{X} = \mathbb{Y} = \Omega = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = \Theta = [1, 2]$, $K(\lambda) = [0, \lambda]$ and $F : \mathbb{R} \times \mathbb{R} \times \Theta \rightrightarrows \mathbb{R}$ defined

$$F(x, y, \mu) = \left\{ -|x-y|^{1/2}, (y-x)\mu \right\}.$$

It is clear that $F(\cdot, \cdot, \mu)$ is nonconvex on $K(\lambda)$ for all $\mu \in \Theta$.

To verify Hölder strong pseudomonotonicity, we assume that for all $x, y \in K(\lambda)$,

$$F(x, y, \mu) \cap C \neq \emptyset.$$

Then, $(y-x)\mu > 0$, i.e., $(x-y)\mu < 0$, and so

$$(x-y)\mu = -|y-x|\mu < -|y-x| < -|x-y|^{1/2} < 0.$$

This leads to

$$(x-y)\mu + |x-y|^{1/2} < 0 \text{ and } -|x-y|^{1/2} + |x-y|^{1/2} = 0.$$

Hence, $F(y, x, \mu) + |x-y|^{1/2}.1 \subseteq -C$. Therefore, F satisfies the condition for Hölder strong pseudomonotonicity with respect to C.

To check quasimonotonicity with respect to C, we assume that for any $x, y \in K(\lambda)$,

$$F(x, y, \mu) \subseteq \mathbb{Y} \setminus C.$$

This means that $F(x, y, \mu) \subseteq]-\infty, 0[$, it points out $(y-x)\mu < 0$, or equivalently

$$(x-y)\mu > 0.$$

Thus, $F(y, x, \mu) \not\subseteq \mathbb{Y} \setminus C$. It is easy to see that F is upper globally Hölder continuous in the second and third arguments on $K(\Lambda) \times \Theta$. Then, all assumptions of Theorem 3.4 hold true, and so $\mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})(\cdot, \cdot)$ is globally Hölder continuous on Λ.

3.2. **Convex set-valued equilibrium problems.** In this subsection, we utilize convexity conditions to study Hölder/Lipschitz continuity for approximate efficient solution maps of equilibrium problems.

For each $(\varepsilon, \lambda, \mu) \in \mathbb{R}_+ \times \Lambda \times \Theta$, we define the approximate efficient solution set of (SEP) as follows.

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon, \lambda, \mu) := \{x \in K(\lambda) \mid F(x, y, \mu) \cap (\varepsilon \mathbb{B}_{\mathbb{Y}} + C) \neq \emptyset \ \text{ for all } y \in K(\lambda)\}.$$

Then, by the definition, we can verify that

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(0, \lambda, \mu) = \mathrm{Sol}_{\mathrm{E}}(\mathrm{SEP})(\lambda, \mu), \text{ and}$$
$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda, \mu) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu) \text{ for all } 0 \leq \varepsilon_1 \leq \varepsilon_2.$$

Theorem 3.7. *Let $\varepsilon_0 > 0$ be a given point. For a given vector $(\lambda, \mu) \in \Lambda \times \Theta$, assume that*

(i) *$K(\lambda)$ is bounded and convex;*
(ii) *F is lower $(-C)$-quasiconvex in the first variable on $K(\lambda)$.*

Then, $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\cdot, \lambda, \mu)$ is globally Lipschitz continuous on $[\varepsilon_0, +\infty[$.

Proof. Let $\varepsilon_1, \varepsilon_2 \in [\varepsilon_0, +\infty[$ be arbitrary with $\varepsilon_1 \leq \varepsilon_2$. Take any $x_2 \in \widetilde{\mathrm{Sol}}_{\mathrm{E}}$ $(\mathrm{SEP})(\varepsilon_2, \lambda, \mu)$ and $x_0 \in \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(0, \lambda, \mu)$, then we have

$$F(x_0, y, \mu) \cap C \neq \emptyset \text{ and } F(x_2, y, \mu) \cap (\varepsilon_2 \mathbb{B}_{\mathbb{Y}} + C) \neq \emptyset \quad \text{ for all } y \in K(\lambda),$$

that is,

$$0 \in F(x_0, y, \mu) - C \text{ and } 0 \in F(x_2, y, \mu) + \varepsilon_2 \mathbb{B}_{\mathbb{Y}} - C.$$

Consequently,

$$0 \in \frac{\varepsilon_1}{\varepsilon_2} F(x_2, y, \mu) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} F(x_0, y, \mu) + \varepsilon_1 \mathbb{B}_{\mathbb{Y}} - C.$$

Equivalently,

$$\left(\frac{\varepsilon_1}{\varepsilon_2} F(x_2, y, \mu) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} F(x_0, y, \mu)\right) \cap (\varepsilon_1 \mathbb{B}_{\mathbb{Y}} + C) \neq \emptyset. \tag{8}$$

Since $K(\lambda)$ is convex, we get

$$x_1 := \frac{\varepsilon_1}{\varepsilon_2} x_2 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} x_0 \in K(\lambda).$$

By the lower $(-C)$-quasiconvexity of $F(\cdot, y, \mu)$ and (8), Remark 2.7(a) derives that

$$F(x_1, y, \mu) \cap (\varepsilon_1 \mathbb{B}_{\mathbb{Y}} + C) \neq \emptyset \quad \text{ for all } y \in K(\lambda).$$

Then,

$$x_1 \in \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda, \mu).$$

It is clear that

$$\|x_2 - x_1\| = \frac{|\varepsilon_1 - \varepsilon_2|}{\varepsilon_2} \|x_2 - x_0\|.$$

Combining this with (i), we can find $\rho > 0$ such that

$$\|x_2 - x_1\| \leq \frac{\rho}{\varepsilon_0} |\varepsilon_1 - \varepsilon_2|.$$

Therefore,

$$x_2 \in x_1 + \frac{\rho}{\varepsilon_0} |\varepsilon_1 - \varepsilon_2| \mathbb{B}_{\mathbb{X}},$$

where $\mathbb{B}_{\mathbb{X}}$ is the closed unit ball in $\mathbb{X}$. Consequently,

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2,\lambda,\mu) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1,\lambda,\mu) + \frac{\rho}{\varepsilon_0}|\varepsilon_1-\varepsilon_2|\mathbb{B}_{\mathbb{X}}. \tag{9}$$

On the other hand, since $\varepsilon_1 \leq \varepsilon_2$, we have

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1,\lambda,\mu) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2,\lambda,\mu),$$

and so

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1,\lambda,\mu) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2,\lambda,\mu)+\frac{\rho}{\varepsilon_0}|\varepsilon_1-\varepsilon_2|\mathbb{B}_{\mathbb{X}} \quad \text{for all } \varepsilon_1,\varepsilon_2 \in [\varepsilon_0,+\infty[.$$

This together with (9) implies that the map $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\cdot,\lambda,\mu)$ is globally Lipschitz continuous on $[\varepsilon_0,+\infty[$. □

Remark 3.8. In Theorem 3.7, we employ lower cone-quasiconvexity to establish the Lipschitz continuity of the approximate solution map. By leveraging this technique, we eliminate the need for the Lipschitz conditions on constrained maps required in Lemma 3.1 of [15] and Lemma 3.3 of [14]. Therefore, Theorem 3.7 can be regarded as an improved version of these lemmas.

Theorem 3.9. *Let $\varepsilon_0 > 0$ be a given point. For a fixed point $\lambda \in \Lambda$, assume that*
(i) *$K(\lambda)$ is bounded and convex;*
(ii) *F is lower $(-C)$-quasiconvex in the first variable on $K(\lambda)$;*
(iii) *F is upper globally Lipschitz continuous in the third variable on Θ.*

Then, $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\cdot,\lambda,\cdot)$ is globally Lipschitz continuous on $[\varepsilon_0,+\infty[\times\Theta$.

Proof. In view of (i) and (ii), Theorem 3.7 gives us to find $\rho > 0$ such that for all $\mu \in \Theta$,

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1,\lambda,\mu) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2,\lambda,\mu)+\frac{\rho}{\varepsilon_0}|\varepsilon_1-\varepsilon_2|\mathbb{B}_{\mathbb{X}} \quad \text{for all } \varepsilon_1,\varepsilon_2 \in [\varepsilon_0,+\infty[. \tag{10}$$

It follows from (iii) that for all $x, y \in K(\lambda)$, there exists $\ell > 0$ such that

$$F(x,y,\mu_1) \subseteq F(x,y,\mu_2) + \ell\|\mu_1-\mu_2\|\mathbb{B}_{\mathbb{Y}} - C \quad \text{for all } \mu_1,\mu_2 \in \Theta. \tag{11}$$

For $(\varepsilon_1,\mu_1),(\varepsilon_2,\mu_2) \in [\varepsilon_0,+\infty[\times\Theta$, we set

$$r := \ell\|\mu_1-\mu_2\| \text{ and } \tau := \varepsilon_2-\varepsilon_0,$$

and consider two cases.

Case 1. If $r \leq \tau$, then $\varepsilon_2 - r \geq \varepsilon_2 - \tau = \varepsilon_0$. Let $\bar{x} \in \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2-r,\lambda,\mu_1)$ be arbitrary. Then, for all $y \in K(\lambda)$, we have

$$F(\bar{x},y,\mu_1) \cap ((\varepsilon_2-r)\mathbb{B}_{\mathbb{Y}} + C) \neq \emptyset.$$

Consequently, for each $y \in K(\lambda)$, there exists some element $z_1 \in F(\bar{x},y,\mu_1)$ such that

$$z_1 \in (\varepsilon_2-r)\mathbb{B}_{\mathbb{Y}} + C. \tag{12}$$

Thanks to (11), we obtain

$$z_1 \in F(\bar{x},y,\mu_2) + \ell\|\mu_1-\mu_2\|\mathbb{B}_{\mathbb{Y}} - C = F(\bar{x},y,\mu_2) + r\mathbb{B}_{\mathbb{Y}} - C.$$

It means that there exist $z_2 \in F(\bar{x},y,\mu_2)$ and $c_1 \in C$ such that $z_1 - z_2 + c_1 \in r\mathbb{B}_{\mathbb{Y}}$, or equivalently

$$z_2 - z_1 - c_1 \in r\mathbb{B}_{\mathbb{Y}}.$$

Combining this with (12), we get

$$\begin{aligned} z_2 &\in z_1 + c_1 + r\mathbb{B}_{\mathbb{Y}} \\ &\in (\varepsilon_2 - r)\mathbb{B}_{\mathbb{Y}} + C + C + r\mathbb{B}_{\mathbb{Y}} \subseteq \varepsilon_2\mathbb{B}_{\mathbb{Y}} + C. \end{aligned}$$

Therefore,

$$F(\bar{x}, y, \mu_2) \cap (\varepsilon_2\mathbb{B}_{\mathbb{Y}} + C) \neq \emptyset,$$

and thus $\bar{x} \in \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2)$. Equivalently,

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2 - r, \lambda, \mu_1) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2).$$

Then, by Theorem 3.7, we achieve

$$\begin{aligned} \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda, \mu_1) &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_1) + \frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2|\mathbb{B}_{\mathbb{X}} \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2 - r, \lambda, \mu_1) + \frac{\rho}{\varepsilon_0}\left(r + |\varepsilon_1 - \varepsilon_2|\right)\mathbb{B}_{\mathbb{X}} \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2) + \frac{\rho}{\varepsilon_0}\left(r + |\varepsilon_1 - \varepsilon_2|\right)\mathbb{B}_{\mathbb{X}} \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2) + \frac{\rho}{\varepsilon_0}\left(\ell\|\mu_1 - \mu_2\| + |\varepsilon_1 - \varepsilon_2|\right)\mathbb{B}_{\mathbb{X}}. \end{aligned}$$

Similarly, we also have

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda, \mu_1) + \frac{\rho}{\varepsilon_0}\left(\ell\|\mu_1 - \mu_2\| + |\varepsilon_1 - \varepsilon_2|\right)\mathbb{B}_{\mathbb{X}}.$$

Case 2. If $r > \tau$, then there is a natural number n_0 satisfying

$$\frac{r}{n_0} = \frac{\ell\|\mu_1 - \mu_2\|}{n_0} \leq \tau.$$

Let $\mathbb{P}$ be a partition of segment $[\mu_1, \mu_2]$ with $n_0 + 1$ nodes $u_1, u_2, ..., u_{n_0+1}$ such that

$$u_1 = \mu_1, u_{n_0+1} = \mu_2, \|u_i - u_{i+1}\| = \frac{\|\mu_1 - \mu_2\|}{n_0} \leq \frac{\tau}{\ell}.$$

Thus,

$$\ell\|u_i - u_{i+1}\| \leq \tau.$$

Applying Case 1, we obtain

$$\begin{aligned} \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, u_i) &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, u_{i+1}) + \frac{\rho\ell}{\varepsilon_0}\|u_i - u_{i+1}\|\mathbb{B}_{\mathbb{X}} \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, u_{i+1}) + \frac{\rho\ell}{n_0\varepsilon_0}\|\mu_1 - \mu_2\|\mathbb{B}_{\mathbb{X}}. \end{aligned}$$

Consequently,

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_1) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2) + \frac{\rho\ell}{\varepsilon_0}\|\mu_1 - \mu_2\|\mathbb{B}_{\mathbb{X}}.$$

Then, by Case 1, we get

$$\begin{aligned} \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda, \mu_1) &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_1) + \frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2|\mathbb{B}_{\mathbb{X}} \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2) + \frac{\rho}{\varepsilon_0}\left(\ell\|\mu_1 - \mu_2\| + |\varepsilon_1 - \varepsilon_2|\right)\mathbb{B}_{\mathbb{X}}. \end{aligned}$$

Similarly, we also get

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda, \mu_2) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda, \mu_1) + \frac{\rho}{\varepsilon_0}\left(\ell\|\mu_1 - \mu_2\| + |\varepsilon_1 - \varepsilon_2|\right)\mathbb{B}_{\mathbb{X}}.$$

Therefore, both of two cases, we always have

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_i,\lambda,\mu_i)\subseteq\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_j,\lambda,\mu_j)+\frac{\rho}{\varepsilon_0}\left(\ell\|\mu_i-\mu_j\|+|\varepsilon_i-\varepsilon_j|\right)\mathbb{B}_{\mathbb{X}},$$

for $i,j\in\{1,2\}$, completing the proof. □

We now recall a concept playing an important role in our analysis.

Definition 3.10. (see [8]) For a given number $\rho>0$, the map $G:\mathbb{X}\rightrightarrows\mathbb{Y}$ is termed to satisfied the *ρ-uniformly bounded diameter* property on $\Omega\subseteq\mathbb{X}$ if for all $x\in\Omega$, $\operatorname{diam}G(x)\leq\rho$ where $\operatorname{diam}(\cdot)$ is the diameter of "$\cdot$".

Next, we consider the globally Lipschitz continuity of the efficient solution map to (SEP) under both data perturbation.

Theorem 3.11. *Let $\varepsilon_0>0$ be a given point. Assume that*

(i) *K is globally Lipschitz continuous on Λ;*
(ii) *K has uniformly bounded diameter and convex values on Λ;*
(iii) *F is upper globally Lipschitz continuous on $K(\Lambda)\times K(\Lambda)\times\Theta$;*
(iv) *for $\lambda\in\Lambda$, the map F is lower $(-C)$-quasiconvex in the first component on $K(\lambda)$.*

Then, $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})$ is globally Lipschitz continuous on $[\varepsilon_0,+\infty[\times\Lambda\times\Theta$.

Proof. In view of (i), we can find $\hbar>0$ such that

$$K(\lambda_1)\subseteq K(\lambda_2)+\hbar\|\lambda_1-\lambda_2\|\mathbb{B}_{\mathbb{X}}\quad\text{for all }\lambda_1,\lambda_2\in\Lambda.\tag{13}$$

By the upper globally Lipschitz continuity of F on $K(\Lambda)\times K(\Lambda)\times\Theta$, there exists $\ell>0$ such that for all $(x_1,y_1,\mu_1),(x_2,y_2,\mu_2)\in K(\Lambda)\times K(\Lambda)\times\Theta$,

$$F(x_1,y_1,\mu_1)\subseteq F(x_2,y_2,\mu_2)+\ell\left(\|\bar{x}_1-x_2\|+\|y_1-y_2\|+\|\mu_1-\mu_2\|\right)\mathbb{B}_{\mathbb{Y}}-C.\tag{14}$$

For $(\varepsilon_1,\lambda_1,\mu_1)$, $(\varepsilon_2,\lambda_2,\mu_2)\in[\varepsilon_0,+\infty[\,\times\Lambda\times\Theta$, we set

$$r:=2\ell\hbar\|\lambda_1-\lambda_2\|,\, s:=\ell\|\mu_1-\mu_2\|\text{ and }\tau:=\varepsilon_2-\varepsilon_0,$$

and consider two cases.

Case 1. If $r+s\leq\tau$, then $\varepsilon_2-r-s\geq\varepsilon_2-\tau=\varepsilon_0$. Let $\bar{x}\in\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2-r-s,\lambda_1,\mu_1)$ be arbitrary. Since $\bar{x}\in K(\lambda_1)$, Assumption (i) implies that there exists $x_{22}\in K(\lambda_2)$,

$$\|\bar{x}-x_{22}\|\leq\hbar\|\lambda_1-\lambda_2\|.\tag{15}$$

We claim that $x_{22}\in\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2,\lambda_2,\mu_2)$. Taking any $y_2\in K(\lambda_2)$, there exists $y_1\in K(\lambda_1)$ satisfying

$$\|y_1-y_2\|\leq\hbar\|\lambda_1-\lambda_2\|.\tag{16}$$

It follows from $\bar{x}\in\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2-r-s,\lambda_1,\mu_1)$ and $y_1\in K(\lambda_1)$ that there is $z_1\in F(\bar{x},y_1,\mu_1)$ such that

$$z_1\in(\varepsilon_2-r-s)\mathbb{B}_{\mathbb{Y}}+C.\tag{17}$$

Thanks to (iii), (15) and (16), there exists $\ell>0$ such that

$$\begin{aligned}F(\bar{x},y_1,\mu_1)&\subseteq F(x_{22},y_2,\mu_2)+\ell\left(\|\bar{x}-x_{22}\|+\|y_1-y_2\|+\|\mu_1-\mu_2\|\right)\mathbb{B}_{\mathbb{Y}}-C\\&\subseteq F(x_{22},y_2,\mu_2)+\left(2\ell\hbar\|\lambda_1-\lambda_2\|+\ell\|\mu_1-\mu_2\|\right)\mathbb{B}_{\mathbb{Y}}-C\\&\subseteq F(x_{22},y_2,\mu_2)+(r+s)\mathbb{B}_{\mathbb{Y}}-C.\end{aligned}$$

For $z_1\in F(\bar{x},y_1,\mu_1)$, there exist $z_2\in F(x_{22},y_2,\mu_2)$ and $c_1\in C$ such that

$$z_1-z_2+c_1\in(r+s)\mathbb{B}_{\mathbb{Y}},$$

and consequently

$$z_2 - z_1 - c_1 \in (r+s)\mathbb{B}_{\mathbb{Y}}.$$

Combining this with (17), we get

$$\begin{aligned} z_2 &\in z_1 + c_1 + (r+s)\mathbb{B}_{\mathbb{Y}} \\ &\in (\varepsilon_2 - r - s)\mathbb{B}_{\mathbb{Y}} + C + C + (r+s)\mathbb{B}_{\mathbb{Y}} \subseteq \varepsilon_2\mathbb{B}_{\mathbb{Y}} + C. \end{aligned}$$

Therefore,

$$F(x_{22}, y_2, \mu_2) \cap (\varepsilon_2\mathbb{B}_{\mathbb{Y}} + C) \neq \emptyset \quad \text{ for all } y \in K(\lambda_2),$$

and so $x_{22} \in \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2)$.

On the other hand, in view of Theorem 3.7, we have

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda_1, \mu_1) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_1, \mu_1) + \frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2|\mathbb{B}_{\mathbb{X}}, \tag{18}$$

and

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_1, \mu_1) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2 - r - s, \lambda_1, \mu_1) + \frac{(r+s)\rho}{\varepsilon_0}\mathbb{B}_{\mathbb{X}}. \tag{19}$$

It follows (15) that

$$\bar{x} \in x_{22} + \hbar\|\lambda_1 - \lambda_2\|\mathbb{B}_{\mathbb{X}}.$$

Thus,

$$\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2 - r - s, \lambda_1, \mu_1) \subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2) + \hbar\|\lambda_1 - \lambda_2\|\mathbb{B}_{\mathbb{X}}. \tag{20}$$

Combining (18), (19) and (20), one has

$$\begin{aligned} &\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda_1, \mu_1) \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_1, \mu_1) + \frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2|\mathbb{B}_{\mathbb{X}} \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2 - r - s, \lambda_1, \mu_1) + \frac{\rho}{\varepsilon_0}\left(r + s + |\varepsilon_1 - \varepsilon_2|\right)\mathbb{B}_{\mathbb{X}} \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2) + \left[\frac{\rho}{\varepsilon_0}\left(r + s + |\varepsilon_1 - \varepsilon_2|\right) + \hbar\|\lambda_1 - \lambda_2\|\right]\mathbb{B}_{\mathbb{X}}, \end{aligned}$$

and hence

$$\begin{aligned} &\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda_1, \mu_1) \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2) \\ &\quad + \left[\frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2| + \left(\frac{\rho}{\varepsilon_0}2\ell\hbar + \hbar\right)\|\lambda_1 - \lambda_2\| + \frac{\rho}{\varepsilon_0}\ell\|\mu_1 - \mu_2\|\right]\mathbb{B}_{\mathbb{X}}. \end{aligned}$$

Similarly, we also obtain

$$\begin{aligned} &\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2) \\ &\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda_1, \mu_1) \\ &\quad + \left[\frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2| + \left(\frac{\rho}{\varepsilon_0}2\ell\hbar + \hbar\right)\|\lambda_1 - \lambda_2\| + \frac{\rho}{\varepsilon_0}\ell\|\mu_1 - \mu_2\|\right]\mathbb{B}_{\mathbb{X}}. \end{aligned}$$

Case 2. If $r+s > \tau$, then there is a natural number n_0 satisfying $\frac{1}{n_0} \leq \min\{\frac{\tau}{2r}, \frac{\tau}{2s}\}$. Let $\mathbb{P}$ be a partition of segment $[\lambda_1, \lambda_2]$ with $n_0 + 1$ nodes $u_1, u_2, ..., u_{n_0+1}$ such that

$$u_1 = \lambda_1, u_{n_0+1} = \lambda_2, \|u_i - u_{i+1}\| = \frac{\|\lambda_1 - \lambda_2\|}{n_0}.$$

Then,

$$\|u_i - u_{i+1}\| \leq \frac{\|\lambda_1 - \lambda_2\|}{n_0} \leq \frac{\tau}{2\ell\hbar},$$

that is,

$$2\ell\hbar\|u_i - u_{i+1}\| \leq \frac{\tau}{2}. \tag{21}$$

In addition, let $\mathbb{V}$ be a partition of segment $[\mu_1, \mu_2]$ with n_0+1 nodes $v_1, v_2, ..., v_{n_0+1}$ such that

$$v_1 = \mu_1, v_{n_0+1} = \mu_2, \|v_i - v_{i+1}\| = \frac{\|\mu_1 - \mu_2\|}{n_0}.$$

It is clear that

$$\|v_i - v_{i+1}\| \leq \frac{\|\mu_1 - \mu_2\|}{n_0} \leq \frac{\tau}{2\ell},$$

and so,

$$\ell\|v_i - v_{i+1}\| \leq \frac{\tau}{2}. \tag{22}$$

From (21) and (22),one has

$$2\ell\hbar\|u_i - u_{i+1}\| + \ell\|v_i - v_{i+1}\| \leq \tau.$$

Applying Case 1, we get

$$\begin{aligned}
&\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, u_i, v_i)\\
&\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, u_{i+1}, v_{i+1}) + \left[\left(\frac{\rho}{\varepsilon_0}2\ell\hbar + \hbar\right)\|u_i - u_{i+1}\| + \frac{\rho}{\varepsilon_0}\ell\|v_i - v_{i+1}\|\right]\mathbb{B}_{\mathbb{X}}\\
&\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, u_{i+1}, v_{i+1}) + \left[\left(\frac{\rho}{\varepsilon_0}2\ell\hbar + \hbar\right)\frac{\|\lambda_1 - \lambda_2\|}{n_0} + \frac{\rho}{\varepsilon_0}\ell\frac{\|\mu_1 - \mu_2\|}{n_0}\right]\mathbb{B}_{\mathbb{X}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_1, \mu_1)\\
&\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2) + \left[\left(\frac{\rho}{\varepsilon_0}2\ell\hbar + \hbar\right)\|\lambda_1 - \lambda_2\| + \frac{\rho}{\varepsilon_0}\ell\|\mu_1 - \mu_2\|\right]\mathbb{B}_{\mathbb{X}}.
\end{aligned}$$

Then, by using Case 1 again, we get

$$\begin{aligned}
&\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda_1, \mu_1)\\
&\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_1, \mu_1) + \frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2|\mathbb{B}_{\mathbb{X}}\\
&\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2)\\
&\quad + \left[\frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2| + \left(\frac{\rho}{\varepsilon_0}2\ell\hbar + \hbar\right)\|\lambda_1 - \lambda_2\| + \frac{\rho}{\varepsilon_0}\ell\|\mu_1 - \mu_2\|\right]\mathbb{B}_{\mathbb{X}}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_2, \lambda_2, \mu_2)\\
&\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_1, \lambda_1, \mu_1)\\
&\quad + \left[\frac{\rho}{\varepsilon_0}|\varepsilon_1 - \varepsilon_2| + \left(\frac{\rho}{\varepsilon_0}2\ell\hbar + \hbar\right)\|\lambda_1 - \lambda_2\| + \frac{\rho}{\varepsilon_0}\ell\|\mu_1 - \mu_2\|\right]\mathbb{B}_{\mathbb{X}}.
\end{aligned}$$

Therefore, both of two cases, we always have

$$\begin{aligned}
&\widetilde{\mathrm{Sol}}(\mathrm{SEP})(\varepsilon_i, \lambda_i, \mu_i)\\
&\subseteq \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon_j, \lambda_j, \mu_j)
\end{aligned}$$

$$+\left[\frac{\rho}{\varepsilon_0}|\varepsilon_i-\varepsilon_j|+\left(\frac{\rho}{\varepsilon_0}2\ell\hbar+\hbar\right)\|\lambda_i-\lambda_j\|+\frac{\rho}{\varepsilon_0}\ell\|\mu_i-\mu_j\|\right]\mathbb{B}_{\mathbb{X}}.$$

for $i,j\in\{1,2\}$, completing the proof. □

The following example illustrates that Theorem 3.11 is applicable.

Example 3.12. Let $\mathbb{X}=\mathbb{W}=\mathbb{R}, \mathbb{Y}=\mathbb{R}^2$, $C=\mathbb{R}_+\times\{0\}$, $\Lambda=\Theta=[0,1], K(\lambda)=[\lambda-2,3]$ and

$$F(x,y,\mu)=\operatorname{conv}\{(x,x),(x,x^2-6)\}+(\mu+|y|)\mathbb{B}_{\mathbb{Y}}.$$

Find $x\in K(\lambda)$ such that

$$F(x,y,\mu)\cap C\neq\emptyset \quad \text{for all } y\in K(\lambda)$$

Assumption (i) of Theorem 3.11 holds true with $\hbar=1$. Obviously, Assumption (ii) of Theorem 3.11 is true with $\rho=5$. Moreover, for any $x_1,x_2\in K(\lambda)$ and $t\in[0,1]$, one has

$$\begin{aligned}
&(1-t)F(x_1,y,\mu)+tF(x_2,y,\mu)\\
&=(1-t)\operatorname{conv}\{(x_1,x_1),(x_1,x_1^2-6)\}+t\operatorname{conv}\{(x_2,x_2),(x_2,x_2^2-6)\}+(\mu+|y|)\mathbb{B}_{\mathbb{Y}}\\
&=\operatorname{conv}\{(1-t)(x_1,x_1)+t(x_2,x_2),(1-t)(x_1,x_1^2-6)+t(x_2,x_2^2-6)\}+\mu^y\mathbb{B}_{\mathbb{Y}}\\
&=\operatorname{conv}\{(x_t,x_t),(x_t,(1-t)x_1^2+tx_2^2-6)\}+(\mu+|y|)\mathbb{B}_{\mathbb{Y}}\\
&\subseteq\operatorname{conv}\{(x_t,x_t),(x_t,x_t^2-6)\}+(\mu+|y|)\mathbb{B}_{\mathbb{Y}}-C=F(x_t,y,\mu)-C,
\end{aligned}$$

where $x_t:=(1-t)x_1+tx_2$. Then, $F(\cdot,y,\mu)$ is lower $(-C)$-convex on $K(\lambda)$.

Moving to the global Lipschitz continuity of F, for $x_1,y_1,x_2,y_2\in[-2,3]$ and $\mu_1,\mu_2\in[0,1]$, we have

$$\operatorname{conv}\{(x_1,x_1),(x_1,x_1^2-6)\}\subseteq\operatorname{conv}\{(x_2,x_2),(x_2,x_2^2-6)\}+\sqrt{37}|x_1-x_2|\mathbb{B}_{\mathbb{Y}},$$

and

$$(\mu_1+|y_1|)\mathbb{B}_{\mathbb{Y}}\subseteq(\mu_2+|y_2|)\mathbb{B}_{\mathbb{Y}}+(|\mu_1-\mu_2|+|y_1-y_2|)\,\mathbb{B}_{\mathbb{Y}}.$$

Therefore, by choosing $\ell=\sqrt{37}$, Assumptions (iii) and (iv) of Theorem 3.11 hold true. Then, applying Theorem 3.11, the solution map $\widetilde{\operatorname{Sol}}_{\mathrm{E}}(\mathrm{SEP})$ is globally Lipschitz continuous on $[\varepsilon_0,+\infty[\times\Lambda\times\Theta$.

Remark 3.13. - Existing works have demonstrated that analyzing the Lipschitz properties of the efficient solution maps of equilibrium problems through a direct approach is challenging, primarily because the complement of the cone K is not closed under vector addition. Consequently, most studies on this topic adopt an indirect approach, specifically through scalarization methods. Although this approach is quite effective, it requires additional conditions for the scalar representation of the solution set of the vector problem model under consideration. For instance, convexity conditions on objective maps are needed for linear scalarization functions [30, 40], and cone solidness is required for a general Gerstewitz nonlinear scalar function [14,15]. By contrast, using the direct approach presented in Theorem 3.11, we establish the global Lipschitz property without assuming any additional conditions. Therefore, Theorem 3.11 contributes not only a new result but also introduces new techniques.

- Recently, in [8], the authors studied the Lipschitz conditions of the solution map of a single-valued strong equilibrium problem in a reflexive Banach space. By combining the weak compactness of the closed unit ball with a general Hiriart-Urruty oriented distance function, they established the Lipschitz continuity of approximate solution maps for the reference problem. In this context, although the convexity condition imposed in Theorem 3.11 appears slightly stronger than the corresponding condition in Theorem 2 of [8], our approach allows us to address the problem in incomplete normed spaces.

4. **Applications.** Motivated by the interesting applications of equilibrium problems in studying the properties of solutions to optimal control problems, as proposed in the excellent papers [24–26], in this section, we will apply the results obtained in the previous section to analyze the stability of two optimal control models in biology and economics.

4.1. **Lotka−Volterra model.** The predator-prey system was independently developed by Lotka [35] and Volterra [44] and has since been known as the Lotka–Volterra system, which describes the interaction between a predator and its prey [37]. For a finite time interval $[0,T]$, for each $t \in [0,T]$, let $\zeta(t)$ and $\xi(t)$ be the number of the individuals of the prey population and the predator one, respectively. A prey-predator system is defined by

$$\dot{\zeta}(t) = \zeta(t)\left(a_1 - b_1\xi(t)\right), \tag{23}$$

$$\dot{\xi}(t) = \xi(t)\left(a_2\zeta(t) - b_2\right), \tag{24}$$

where a_1: the natural growth rate of the prey (in the absence of predators); b_1: the rate at which prey are killed by predators; a_2: the growth rate of the predator (dependent on the prey population); b_2: the natural death rate of the predator (in the absence of prey).

Suppose now that hunter populations are introduced in the ecosystem and their acts both on the preys and predators. At each moment t, the number of the hunted individuals is assumed to be proportional to the total number of the existing predators. For a finite time interval $[0,T]$ and functions $u_1, u_2 : [0,T] \to [0,1]$, we consider the control map $u : [0,T] \to [0,1] \times [0,1]$ defined by $u(t) := (u_1(t), u_2(t))$. Thus, the dynamics of the new ecosystem is described by the system of ordinary differential equations

$$\dot{\zeta}(t) = \zeta(t)\left(a_1 - b_1\xi(t) - c_1u_1(t)\right), \tag{25}$$

$$\dot{\xi}(t) = \xi(t)\left(a_2\zeta(t) - c_2u_2(t) - b_2\right), \tag{26}$$

$$\min\{\zeta(0), \xi(0), \zeta(T), \xi(T)\} > 0, \tag{27}$$

where the constants c_1, c_2 represent the maximum level of hunting in a time instant for each population. By setting

$$\kappa := (\zeta, \xi) \text{ and } g(\kappa, u) := \left(\zeta(t)\left(a_1 - b_1\xi(t) - c_1u_1(t)\right), \xi(t)\left(a_2\zeta(t) - c_2u_2(t) - b_2\right)\right),$$

then the system (25)−(27) is rewritten as follows.

$$\dot{\kappa} = g(\kappa, u) \text{ and } \min\{\zeta(0), \xi(0), \zeta(T), \xi(T)\} > 0. \tag{28}$$

Definition 4.1. [1, page 27] A map $v : [0,T] \to \mathbb{R}^2$ that is measurable on $[0,T]$ is said to be *essentially bounded* on $[0,T]$ if there is a constant M such that

$$\|v(t)\| \leq M \quad \text{a.e. on} \quad [0,T].$$

The greatest lower bound of such constants M is called *the essential supremum of* $\|v\|$ on $[0,T]$, and is denoted by $\operatorname{ess\,sup}_{t\in[0,T]} \|v(t)\|$. We denote by $L^\infty([0,T], \mathbb{R}^2)$ the vector space of all maps u that are essentially bounded on $[0,T]$, maps being once again identified if they are equal a.e. on $[0, T]$. It is easily checked that the functional $\|\cdot\|_\infty$ defined by

$$\|\upsilon\|_\infty = \operatorname*{ess\,sup}_{t\in[0,T]} \|\upsilon(t)\|$$

is a norm on $L^\infty([0,T], \mathbb{R}^2)$.

We denote

$$\mathcal{K} := \{u \in L^\infty([0,T], \mathbb{R}^2) : u(t) \in [0,1] \times [0,1] \text{ for all } t \in [0,T]\}.$$

Motivated by [43, 45], we consider the static optimal control problem (SOCP)

$$\max\left\{\mu_1\zeta(T) + \mu_2\xi(T) : g(\kappa, u) = 0, u \in \mathcal{K}\right\},$$

with nonnegative weights μ_1, μ_2 representing the individual economical values of preys and predators, respectively. Here, $g(\kappa, u) = 0$ represents the set of equilibrium points with respect to the control map u, as defined in [18, 43]. It follows from $g(\kappa, u) = 0$ that

$$\zeta(T) = \frac{c_2 u_2(T) + b_2}{a_2} \text{ and } \xi(T) = \frac{a_1 - c_1 u_1(T)}{b_1}.$$

Then, the problem (SOCP) is rewritten in the following form

$$\max_{u=(u_1,u_2)\in\mathcal{K}} \left(\mu_1 \frac{c_2 u_2(T) + b_2}{a_2} + \mu_2 \frac{a_1 - c_1 u_1(T)}{b_1}\right).$$

To study the Lipschitz property for (SOCP), we examine the convex property of the constraint set $\mathcal{K}$.

Lemma 4.2. *The set $\mathcal{K}$ is convex.*

Proof. Let $u, \bar{u} \in \mathcal{K}$ be arbitrary. We have

$$u, \bar{u} \in L^\infty([0,T], \mathbb{R}^2) \text{ and } u(t), \bar{u}(t) \in [0,1] \times [0,1] \text{ for all } t \in [0,T]\}.$$

Then, for all $s \in [0,1]$, we get

$$(1-s)u + s\bar{u} \in L^\infty([0,T], \mathbb{R}^2) \text{ and } (1-s)u(t) + s\bar{u}(t) \in [0,1]\times[0,1] \text{ for all } t \in [0,T]\},$$

and therefore $(1-s)u + s\bar{u} \in \mathcal{K}$. □

Corollary 4.3. *For any $\varepsilon_0 > 0$, $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SOCP})$ is globally Lipschitz continuous on $[\varepsilon_0, +\infty[\times\mathbb{R}^2_+$.*

Proof. Obviously, for all $\varepsilon > 0$ and $\mu = (\mu_1, \mu_2)$, we have $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SOCP})(\varepsilon, \mu) \equiv \widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SEP})(\varepsilon, \mu)$, where

$$\begin{aligned} F(u, v, \mu) &= \left(\mu_1 \frac{c_2 u_2(T) + b_2}{a_2} + \mu_2 \frac{a_1 - c_1 u_1(T)}{b_1}\right) \\ &\quad - \left(\mu_1 \frac{c_2 v_2(T) + b_2}{a_2} + \mu_2 \frac{a_1 - c_1 v_1(T)}{b_1}\right) \\ &= \frac{c_2\mu_1}{a_2}\left(u_2(T) - v_2(T)\right) + \frac{\mu_2 c_1}{b_1}\left(v_1(T) - u_1(T)\right), \end{aligned}$$

for all $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Thus, to obtain the conclusion of Corollary 4.3, we will show that all the assumptions of Theorem 3.9 are fulfilled. By Lemma 4.2, Assumption (i) in Theorem 3.9 holds true. We now check other assumptions in Theorem 3.9.

$\diamond$ *F is affine in the first component on $\mathcal{K}$*: Let $u = (u_1, u_2), \bar{u} = (\bar{u}_1, \bar{u}_2), v = (v_1, v_2) \in \mathcal{K}$ and $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2_+$ be arbitrary. Then, for all $s \in [0, 1]$,

$$\begin{aligned} F((1-s)u + s\bar{u}, v, \mu) &= \frac{c_2\mu_1}{a_2}\left((1-s)u_2(T) + s\bar{u}_2(T) - v_2(T)\right) \\ &\quad + \frac{\mu_2 c_1}{b_1}\left(v_1(T) - (1-s)u_1(T) - s\bar{u}_1(T)\right) \\ &= (1-s)\left(\frac{c_2\mu_1}{a_2}\left(u_2(T) - v_2(T)\right) + \frac{\mu_2 c_1}{b_1}\left(v_1(T) - u_1(T)\right)\right) \\ &\quad + s\left(\frac{c_2\mu_1}{a_2}\left(u_2(T) - \bar{v}_2(T)\right) + \frac{\mu_2 c_1}{b_1}\left(\bar{v}_1(T) - u_1(T)\right)\right) \\ &= (1-s)F(u, v, \mu) + sF(\bar{u}, v, \mu). \end{aligned}$$

$\diamond$ *F is globally Lipschitz continuous in the third variable on $\mathbb{R}^2_+$.* Let $u = (u_1, u_2), v = (v_1, v_2)$ belong to $\mathcal{K}$ and $\mu = (\mu_1, \mu_2), \bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) \in \mathbb{R}^2_+$, we have

$$F(u, v, \mu) = \frac{c_2\mu_1}{a_2}\left(u_2(T) - v_2(T)\right) + \frac{c_1\mu_2}{b_1}\left(v_1(T) - u_1(T)\right),$$

and

$$F(u, v, \bar{\mu}) = \frac{c_2\bar{\mu}_1}{a_2}\left(u_2(T) - v_2(T)\right) + \frac{c_1\bar{\mu}_2}{b_1}\left(v_1(T) - u_1(T)\right).$$

Then,

$$\begin{aligned} &|F(u, v, \mu) - F(u, v, \bar{\mu})| \\ &\leq \frac{c_2}{a_2}\left|u_2(T) - v_2(T)\right|\left|\mu_1 - \bar{\mu}_1\right| + \frac{c_1}{b_1}\left|v_1(T) - u_1(T)\right|\left|\mu_2 - \bar{\mu}_2\right|. \end{aligned}$$

Combining this with $u, v \in L^\infty([0, T], \mathbb{R}^2)$, there exists $M > 0$ such that

$$\begin{aligned} |F(u, v, \mu) - F(u, v, \bar{\mu})| &\leq \frac{2Mc_2}{a_2}|\mu_1 - \bar{\mu}_1| + \frac{2Mc_1}{b_1}|\mu_2 - \bar{\mu}_2| \\ &\leq \left(\frac{4Mc_2}{a_2} + \frac{4Mc_1}{b_1}\right)\|\mu - \bar{\mu}\|. \end{aligned}$$

Therefore, all assumptions of in Theorem 3.9 hold true, and so $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{SOCP})$ is globally Lipschitz continuous on $[\varepsilon_0, +\infty[\times\mathbb{R}^2_+$. $\square$

4.2. **A cash balance problem.** Businesses always need to maintain a certain amount of cash to ensure their ability to meet daily operational needs, such as payroll, operating expenses, or other payables. However, holding too much cash may cause businesses to miss out on profitable investment opportunities in securities such as bonds, stocks, or other financial instruments. On the other hand, if they hold too little cash, businesses may have to sell securities unexpectedly to meet urgent financial demands. This not only disrupts their investment strategy but also incurs additional brokerage fees and related costs. Therefore, a key challenge for businesses is determining the optimal balance between holding cash and investing in securities to maximize values while minimizing associated costs.

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space with the norm $|\cdot|$. The Banach space $\mathcal{C}([0, T], \mathbb{R}^n)$ is the space of all continuous functions $\psi : [0, T] \to \mathbb{R}^n$ equipped with the norm $\|\psi\| = \max_{t\in[0,T]} |\psi(t)|$. For $1 \leq p \leq \infty$, let $\mathcal{L}^p([0, T], \mathbb{R}^n)$ denote the space of all the Lebesgue integrable functions defined on $[0, T]$ with the norm $\|\cdot\|_p$. For a matrix $A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, the norm of a square matrix A is a non-negative real number defined by

$$\|A\|_\infty := \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|.$$

For a finite time interval $[0,T]$, let $\zeta(t)$ and $\xi(t)$ represent the cash balance and the security balance (both in dollars) at time $t \in [0,T]$, respectively, for a company. Motivated by [28], we now consider parameters r_1, r_2 and r_3 as follows:

◇ r_1 : the interest rates earned on the cash balance,
◇ r_2 : the interest rates earned on the security balance,
◇ r_3 : the instantaneous rate of demand for cash.

Then, the state equations are expressed as:

$$\dot{\zeta}(t) = r_1(t)\zeta(t) - r_3(t) \text{ and } \dot{\xi}(t) = r_2(t)\xi(t) \quad \text{a.e. } t \in [0,T].$$

To ensure the cash balance problem, the company controls the rate of sale of securities $u \in \mathcal{L}^p([0,T],\mathbb{R})$ with $-\upsilon_2 \leq u(t) \leq \upsilon_1$, where υ_1 and υ_2 are nonnegative constants. Then, we reconsider the state equations as follows.

$$\dot{\zeta}(t) = r_1(t)\zeta(t) - r_3(t) + u(t) \quad \text{a.e. } t \in [0,T], \tag{29}$$

and

$$\dot{\xi}(t) = r_2(t)\xi(t) - u(t) \quad \text{a.e. } t \in [0,T]. \tag{30}$$

The objective is to maximize the total value of assets (cash and securities) at the end of the time horizon T, which is expressed by

$$\max\{\zeta(T) + \xi(T)\},$$

subject to (29) and (30).

Let Υ be a nonempty bounded subset of $\mathbb{R}$. Setting $\lambda := (r_1, r_2, r_3)$ is an element of the parameter space Λ which is defined by

$$\Lambda := \left\{\lambda \in \mathcal{L}^q([0,T],\mathbb{R}) \times \mathcal{L}^q([0,T],\mathbb{R}) \times \mathcal{L}^q([0,T],\mathbb{R}) : \lambda(t) \in \Upsilon^3 \quad \text{a.e. } t \in [0,T]\right\}.$$

Let $\mathcal{U}$ be a nonempty convex subset of $\mathcal{L}^p([0,T],\mathbb{R})$, we define

$$\kappa := (\zeta, \xi),\ \mathcal{X} := \mathcal{C}\left([0,T],\mathbb{R}^2\right),\ \mathcal{W} := \mathcal{X} \times \mathcal{U}.$$

Then, (29) and (30) have become

$$\begin{bmatrix} \dot{\zeta}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} r_1(t) & 0 \\ 0 & r_2(t) \end{bmatrix} \begin{bmatrix} \zeta(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} d(t),$$

or equivalently

$$\dot{\kappa}(t) = \begin{bmatrix} r_1(t) & 0 \\ 0 & r_2(t) \end{bmatrix} \kappa(t) + Au(t) + Br_3(t), \tag{31}$$

where $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. For $\lambda = (r_1, r_2, r_3) \in \Lambda$, we define

$$\mathcal{K}(\lambda) := \{w = (\kappa, u) \in \mathcal{X} \times \mathcal{U} : (31) \text{ is satisfied}\},$$

with $\mathcal{K} : \Lambda \rightrightarrows \mathcal{X} \times \mathcal{U}$ is a feasible solution map. Then, the following statement holds true.

Lemma 4.4. *The function $\mathcal{K}$ is globally Lipschitz continuous as well as has uniformly bounded diameter and convex values on Λ.*

Proof. $\diamond$ *For each $\lambda = (r_1, r_2, r_3) \in \Lambda$, the set $\mathcal{K}(\lambda)$ is convex*: Let $w = (\kappa, u)\,, z = (\eta, v) \in \mathcal{K}(\lambda)$ be arbitrary. Since $\mathcal{X}$ and $\mathcal{U}$ are convex,

$$(1-s)w + sz \in \mathcal{X} \times \mathcal{U} \quad \text{for all } s \in [0,1].$$

We get

$$\begin{aligned}
&\frac{d}{dt}\left((1-s)\kappa(t) + \eta(t)\right)\\
&= (1-s)\dot{\kappa}(t) + s\dot{\eta}(t)\\
&= \begin{bmatrix} r_1(t) & 0 \\ 0 & r_2(t) \end{bmatrix} ((1-s)\kappa(t) + s\eta(t)) + A\left((1-s)u(t) + sv(t)\right) + Br_3(t).
\end{aligned}$$

Therefore, $(1-s)w + sz \in \mathcal{K}(\lambda)$ for all $w, z \in \mathcal{K}(\lambda)$, and hence $\mathcal{K}(\lambda)$ is convex.
$\Diamond$ *$\mathcal{K}$ globally Lipschitz continuous on* Λ: For $\lambda_1 = (r_{11}, r_{12}, r_{13}), \lambda_2 = (r_{21}, r_{22}, r_{23}) \in \Lambda$, and $(\kappa_1, u_1) \in \mathcal{K}(\lambda_1)$, we have

$$\dot{\kappa}_1(t) = \begin{bmatrix} r_{11}(t) & 0 \\ 0 & r_{12}(t) \end{bmatrix} \kappa_1(t) + Au_1(t) + Br_{13}(t) \quad \text{a.e. } t \in [0,T]\,. \tag{32}$$

Then, there exists $\kappa_2 \in \mathcal{X}$ such that

$$\dot{\kappa}_2(t) = \begin{bmatrix} r_{21}(t) & 0 \\ 0 & r_{22}(t) \end{bmatrix} \kappa_2(t) + Au_1(t) + Br_{23}(t) \quad \text{a.e. } t \in [0,T]\,. \tag{33}$$

It follows from (32) and (33) that

$$\begin{aligned}
&\dot{\kappa}_1(t) - \dot{\kappa}_2(t)\\
&= \begin{bmatrix} r_{11}(t) & 0 \\ 0 & r_{12}(t) \end{bmatrix} \kappa_1(t) - \begin{bmatrix} r_{21}(t) & 0 \\ 0 & r_{22}(t) \end{bmatrix} \kappa_2(t) + B\left(r_{13}(t) - r_{23}(t)\right)\\
&= \begin{bmatrix} r_{11}(t) & 0 \\ 0 & r_{12}(t) \end{bmatrix} (\kappa_1(t) - \kappa_2(t)) + \begin{bmatrix} r_{11}(t) - r_{21}(t) & 0 \\ 0 & r_{12}(t) - r_{22}(t) \end{bmatrix} \kappa_2(t)\\
&\quad + B\left(r_{13}(t) - r_{23}(t)\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|\dot{\kappa}_1(t) - \dot{\kappa}_2(t)\|\\
&\le \max\{|r_{11}(t)|, |r_{12}(t)|\}\, \|\kappa_1(t) - \kappa_2(t)\| +\\
&\quad + \max\{|r_{11}(t) - r_{21}(t)|\,, |r_{12}(t) - r_{22}(t)|\}\|\kappa_2(t)\| + \|B\|\, |r_{13}(t) - r_{23}(t)|\\
&\le \max\{|r_{11}(t)|\,, |r_{12}(t)|\}\, \|\kappa_1(t) - \kappa_2(t)\| + (\|\kappa_2(t)\| + 1)\, \|\lambda_1 - \lambda_2\|\\
&\le \rho_1\, \|\kappa_1(t) - \kappa_2(t)\| + (\rho_2 + 1)\, \|\lambda_1 - \lambda_2\|,
\end{aligned}$$

where $\max\{|r_{11}(t)|\,, |r_{12}(t)|\} \le \rho_1, \|\kappa_2(t)\| \le \rho_2$, a.e. $t \in [0,T]$,

$$\|\lambda_1 - \lambda_2\| = \max\{|r_{11}(t) - r_{21}(t)|\,, |r_{12}(t) - r_{22}(t)|\,, |r_{13}(t) - r_{23}(t)|\},$$

and

$$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\| := \max\{|a|, |b|, |c|, |d|\}.$$

It follows that

$$\begin{aligned}
\|\kappa_1(T) - \kappa_2(T)\| &\le \int_0^T \left(\rho_1\, \|\kappa_1(s) - \kappa_2(s)\| + (\rho_2 + 1)\, \|\lambda_1 - \lambda_2\|\right) \mathrm{d}s\\
&\le \int_0^T \rho_1\, \|\kappa_1(s) - \kappa_2(s)|\, \mathrm{d}s + T\left(\rho_2 + 1\right) \|\lambda_1 - \lambda_2\|\,.
\end{aligned}$$

Using the Gronwall inequality, we get

$$\begin{aligned}\|\kappa_1(T)-\kappa_2(T)\| &\leq T\,(\rho_2+1)\,\|\lambda_1-\lambda_2\|\,e^{\int_0^T \rho_1 ds}\\ &\leq T\,(\rho_2+1)\,\|\lambda_1-\lambda_2\|\,e^{\rho_1 T}.\end{aligned}$$

Consequently,

$$\|\kappa_1-\kappa_2\| \leq T\,(\rho_2+1)\,e^{\rho_1 T}\,\|\lambda_1-\lambda_2\|\,.$$

It results that

$$\mathcal{K}(\lambda_1) \subseteq \mathcal{K}(\lambda_2) + T\,(\rho_2+1)\,e^{\rho_1 T}\,\|\lambda_1-\lambda_2\|\,\mathbb{B}_{\mathbb{Y}}.$$

Therefore, $\mathcal{K}$ is globally Lipschitz continuous on Λ. □

Next, by setting

$$\varphi(w) := \varphi(\kappa,u) = \zeta(T)+\xi(T) \quad \text{ for all } w=(\kappa,u)\in\mathcal{W}, \kappa=(\zeta,\xi),$$

the cash balance problem (CBP) can be cast as the following problem

$$\max\varphi(w) \text{ subject to } w\in\mathcal{K}(\lambda).$$

The following result is obtained by Theorem 3.11.

Corollary 4.5. *For any* $\varepsilon_0>0$, $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{CBP})$ *is globally Lipschitz continuous on* $[\varepsilon_0,+\infty[\times\Lambda$.

Proof. For $w_1=(\kappa_1,u_1), w_2=(\kappa_2,u_2)\in W$ and $s\in[0,1]$, we have

$$w_s := (1-s)w_1+sw_2 = ((1-s)\kappa_1+s\kappa_2,(1-s)u_1+su_2)\in\mathcal{W}.$$

If $\kappa_i := (\zeta_i,\xi_i)$ for all $i\in\{1,2\}$, then

$$\begin{aligned}\varphi\left(w_s\right) &= \left((1-s)\zeta_1+s\zeta_2\right)(T)+\left((1-s)\xi_1+s\xi_2\right)(T)\\ &= (1-s)\left(\zeta_1(T)+\xi_1(T)\right)+(1-s)\left(\zeta_2(T)+\xi_2(T)\right)\\ &= (1-s)\varphi(w_1)+s\varphi(w_2).\end{aligned}$$

Hence, φ is affine on $\mathcal{W}$. Consequently, the function F defined by

$$F(w,z) := \varphi(w)-\varphi(z) \quad \text{ for all } w,z\in\mathcal{K}(\lambda),$$

is also affine in the first component on $\mathcal{K}(\lambda)$. Combining this with Lemma 4.4, all assumptions in Theorem 3.11 hold true, and so we conclude that $\widetilde{\mathrm{Sol}}_{\mathrm{E}}(\mathrm{CBP})$ is globally Lipschitz continuous on $[\varepsilon_0,+\infty[\times\Lambda$. □

5. **Conclusion.** In this study, we have used a direct approach to investigate the Hölder/Lipschitz properties of the efficient solution map for the vector equilibrium problem with multivalued objective map. With this approach, we have avoided the common additional conditions typically required by indirect methods, such as scalarization of linear or nonlinear functions, as presented in previous works. By applying this approach and the techniques mentioned in the results above, we also establish the stability in the sense of Hölder/Lipschitz continuity of solution maps of two optimal control problems, including one that describes the interaction between a predator and its prey, and another that addresses the balance between holding cash and investing. We believe that with the techniques and approach presented in this paper, along with appropriate adjustments, there is great potential for further application in studying the Hölder/Lipschitz conditions of other optimization models, as well as in practical situations related to optimal models in biology and economics.

Acknowledgments. The authors sincerely thank the editors and reviewers for their valuable remarks and suggestions, which significantly contributed to improving this paper. The first and third authors were funded by Vietnam National Program for the Development of Mathematics $2021-2030$ under grant number B2025-CTT-03. The last author's work was partially supported by Walailak University under the international research collaboration scheme (Contract Number WU-CIA-00310/2025). Part of this work was completed while the first and third authors were visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM), and the authors are grateful for the Institute's hospitality and support.

REFERENCES

[1] R. A. Adams and J. J. Fournier, *Sobolev Spaces*, 2nd edn, vol. 140, Academic Press, New York, 2003.

[2] S. Al-Homidan, Q. H. Ansari and O. Chadli, Noncoercive mixed equilibrium problems under pseudomonotone perturbations and applications to nonlinear evolution equations with lack of coercivity, *Appl. Anal.,* **98** (2019), 1199-1223.

[3] S. Al-Homidan, Q. H. Ansari and O. Chadli, Noncoercive stationary Navier Stokes equations of heat-conducting fluids modeled by hemivariational inequalities: An equilibrium problem approach, *Results Math.,* **74** (2019), 132.

[4] B. Alleche and V. D. Rădulescu, Further on set-valued equilibrium problems and applications to Browder variational inclusions, *J. Optim. Theory Appl.*, **175** (2017), 39-58.

[5] L. Q. Anh, Semicontinuity of the solution maps to vector equilibrium problems with equilibrium constraints, *Optimization*, **71** (2022), 737-751.

[6] L. Q. Anh, P. T. Duoc and T. N. Tam, On Hölder continuity of solution maps to parametric vector primal and dual equilibrium problems, *Optimization*, **67** (2018), 1169-1182.

[7] L. Q. Anh, P. T. Duoc, T. N. Tam and N. C. Thang, Stability analysis for set-valued equilibrium problems with applications to Browder variational inclusions, *Optim. Lett.*, **15** (2021), 613-626.

[8] L. Q. Anh, P. T. Duoc and N. M. Tung, On Lipschitz continuity of solutions to equilibrium problems via the Hiriart-Urruty oriented distance function, *Comp. Applied Math.*, **41** (2022), 57.

[9] L. Q. Anh and P. Q. Khanh, On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems, *J. Math. Anal. Appl.*, **321** (2006), 308-315.

[10] L. Q. Anh and P. Q. Khanh, Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces, *J. Glob. Optim.*, **37** (2007), 449-465.

[11] L. Q. Anh, P. Q. Khanh and D. N. Quy, About semicontinuity of set-valued maps and stability of quasivariational inclusions, *Set-Valued Var. Anal.*, **22** (2014), 533-555.

[12] L. Q. Anh, P. Q. Khanh and T. N. Tam, On Hölder continuity of approximate solutions to parametric equilibrium problems, *Nonlinear Anal.*, **75** (2012), 2293-2303.

[13] L. Q. Anh, V. T. Tai and T. N. Tam, Painlevé-Kuratowski convergences of solutions to nonlinear multiobjective optimal control problems, *Evol. Equ. Control Theory*, **13** (2024), 1229-1249.

[14] L. Q. Anh and T. N. Tam, Sensitivity analysis for parametric vector equilibrium problems, *J. Nonlinear Convex Anal.*, **18** (2017), 1707-1716.

[15] L. Q. Anh, T. N. Tam and N. H. Danh, On Lipschitz continuity of approximate solutions to set-valued equilibrium problems via nonlinear scalarization, *Optimization*, **72** (2023), 439-461.

[16] Q. H. Ansari, E. Köbis and J.-C. Yao, *Vector Variational Inequalities and Vector Optimization: Theory and Applications*, Springer, Cham, 2018.

[17] Q. H. Ansari, P. K. Sharma and N. Hussain, Semi-continuity of the solution maps of set-valued equilibrium problems with equilibrium constraints, *Optimization*, (2024), 1-23.

[18] V. V. Au and D. H. Manh, Chaotic and stable behavior around the equilibrium points of a non-autonomous competitive Lotka-Volterra system, *Discrete Cont. Dynamical Syst.-S*, **18** (2025), 482-498.

[19] M. Balaj and L.-J. Lin, Generalized variational relation problems with applications, *J. Optim. Theory Appl.*, **148** (2011), 1-13.

[20] M. Bianchi and R. Pini, Sensitivity for parametric vector equilibria, *Optimization*, **55** (2006), 221-230.

[21] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Stud.*, **63** (1994), 123-145.

[22] O. Chadli, Q. H. Ansari and S. Al-Homidan, Existence of solutions for nonlinear implicit differential equations: An equilibrium problem approach, *Numer. Funct. Anal. Optim.,* **37** (2016), 1385-1419.

[23] O. Chadli, Q. H. Ansari and S. Al-Homidan, Existence of solutions and algorithms for bilevel vector equilibrium problems: An auxiliary principle technique, *J. Optim. Theory Appl.,* **172** (2017), 726-758.

[24] O. Chadli, Q. H. Ansari, S. Al-Homidan and M. Alshahrani, Optimal control of systems governed by mixed equilibrium problems under monotonicity type conditions with applications, *Appl. Math. Optim.,* **83** (2021), 373-403.

[25] O. Chadli, Q.H. Ansari, S. Al-Homidan, Augmented Lagrangian methods for optimal control problems governed by mixed quasi-equilibrium problems with applications, *Optimal Control Appl. Methods*, **42** (2021), 1178-1205.

[26] O. Chadli, Q. H. Ansari, S. Al-Homidan and M. Alshahrani, Optimal control of problems governed by mixed quasi-equilibrium problems under monotonicity type conditions with applications, *Appl. Math. Optim.,* **83** (2021), 2185-2209.

[27] L. Eslamizadeh and E. Naraghirad, Existence of solutions of set-valued equilibrium problems in topological vector spaces with applications, *Optim. Lett.*, **14** (2020), 65-83.

[28] C. Gaimon, *Optimal Control Theory: Applications to Management Science and Economics*, Springer, Cham, 2002.

[29] N. X. Hai and P. Q. Khanh, Existence of solutions to general quasiequilibrium problems and applications, *J. Optim. Theory Appl.*, **133** (2007), 317-327.

[30] Y. Han, Lipschitz continuity of approximate solution mappings to parametric generalized vector equilibrium problems, *J. Optim. Theory Appl.*, **178** (2018), 763-793.

[31] Y. Han, Nonlinear scalarizing functions in set optimization problems, *Optimization*, **68** (2019), 1685-1718.

[32] S. Jafari, A. P. Farajzadeh, S. Moradi and P. Q. Khanh, Existence results for φ-quasimonotone equilibrium problems in convex metric spaces, *Optimization*, **66** (2017), 293-310.

[33] G. Kassay and V. Rădulescu, *Equilibrium Problems and Applications*, Academic Press, London, 2019.

[34] S. J. Li, X. B. Li, L.N. Wang and K. L. Teo, The Hölder continuity of solutions to generalized vector equilibrium problems, *Eur. J. Oper. Res.*, **199** (2009), 334-338.

[35] A. J. Lotka, *Elements of Physical Biology*, Williams and Wilkins, Baltimore, 1925.

[36] D. T. Luc, E. Sarabi and A. Soubeyran, Existence of solutions in variational relation problems without convexity, *J. Math. Anal. Appl.*, **364** (2010), 544-555.

[37] J. D. Murray, *Mathematical Biology: I. An Introduction*, vol. 17, Springer, New York, 2002.

[38] L. D. Muu and W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, *Nonlinear Anal.*, **18** (1992), 1159-1166.

[39] A. Ruszczyński and A. Shapiro, Optimization of convex risk functions, *Math. Oper. Res.*, **31** (2006), 433-452.

[40] I. Sadeqi and M. Salehi Paydar, Lipschitz continuity of an approximate solution mapping for parametric set-valued vector equilibrium problems, *Optimization*, **65** (2016), 1003-1021.

[41] K. Seto, D. Kuroiwa and N. Popovici, A systematization of convexity and quasiconvexity concepts for set-valued maps, defined by l-type and u-type preorder relations, *Optimization*, **67** (2018), 1077-1094.

[42] T. N. Tam, Hölder continuity of solution maps to parametric set-valued Ky Fan inequalities, *Optimization*, **73** (2024), 623-640.

[43] E. Trélat and E. Zuazua, The turnpike property in finite-dimensional nonlinear optimal control, *J. Differ. Equ.*, **258** (2015), 81-114.

[44] V. Volterra, *Variazioni e Fluttuazioni del Numero d'Individui in Specie Animali Conviventi,*, Società Anonima Tipografica "Leonardo da Vinci", Città di Castello, 1926.

[45] S. Yosida, An optimal control problem of the prey-predator system, *Funck. Ekvacioj*, **25** (1982), 283-293.

[46] W. R. Zame, Competitive equilibria in production economies with an infinite-dimensional commodity space, *Econometrica: J. Econ. Soc.*, **55** (1987), 1075-1108.

Received December 2024; revised February 2025; early access March 2025.