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Qualitative Properties of Robust Benson Efficient Solutions of Uncertain Vector Optimization Problems

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Abstract
In this paper, we consider both unconstrained and constrained uncertain vector optimization problems involving free disposal sets, and study the qualitative properties of their robust Benson efficient solutions. First, we discuss necessary and sufficient optimality conditions for the robust Benson efficient solutions of these problems using the linear scalarization method. Then, by utilizing this approach, we investigate the semicontinuity properties of the solution maps when the problem data is perturbed by parameters given in parameter spaces. Finally, we suggest concepts of approximate robust Benson efficient solutions and investigate Hausdorff well-posedness conditions for such problems with respect to these approximate solutions. Several examples are provided to illustrate the applicability and novelty of the results obtained in this study.

Keywords Uncerntain vector optimization · Optimality condition · Stability · Well-posedness · Robust Benson efficient solution · Free disposal set · Scalarization method

Mathematics Subject Classification 49K27 · 49K40 · 90C31 · 90C46

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1 Introduction

Vector optimization problems play important role in many fields such as politics, business, industrial systems, management science, networks,... When applying vector optimization models to practical situations, depending on the desired outcomes, various concepts of efficient solutions have been considered, such as weak/strong efficient solutions [4, 33], Edgeworth/Pareto efficient solutions [17], Pareto efficient solutions [45], Geoffrion efficient solutions [21], Borwein efficient solutions [10], Benson efficient solutions [8], and Henig efficient solutions [16, 29] among others. Among these, Geoffrion efficient solutions are regarded as having clear practical significance, making [21] the foundational work that sparked subsequent studies on efficient solutions in vector optimization models. Within this line of research, Borwein efficient solutions have emphasized geometric properties and eliminated ineffective decisions in decision-making, and thus establishing themselves as a standard concept in vector optimization [30]. In 1979, Benson [7] introduced the concept of efficient solutions, now known as Benson efficient solutions, which generalize Geoffrion and Borwein efficient solutions and are regarded as a type of efficient solution that meets various requirements in specific optimization models [6–8].

One of the key factors influencing the various types of efficient solutions mentioned above is the structure of the criterion sets used to evaluate solution efficiency. As a result, studying optimization models by replacing or extending the structure of ordering cones has recently become a fascinating and widely explored topic. In this research stream, several sets have been proposed as replacements for the ordering cone, including the improvement set [27, 28, 46, 59, 60], the free disposal set [26, 34, 38, 52, 56, 61], and the co-radiant set [20, 24]. Among these, the free disposal set unifies the concepts of the improvement set and the ordering cone. Consequently, optimization models involving free disposal sets have attracted significant interest from mathematicians in recent years. Let us now provide a brief overview of notable works on optimization problems involving the free disposal set. In 2015, Gutiérrez et al. [26] proposed various concepts of quasi-minimality associated with free disposal sets and described these solutions through scalarization techniques and Lagrange multiplier principles. When specific convexity conditions are met, the findings are derived using linear separation methods and the Fenchel subdifferential. In contrast, for nonconvex problems, the results are formulated using the Gerstewitz nonlinear separation functional and the Mordukhovich subdifferential. Next, in [38], based on the Hiriart-Urruty oriented function, the authors considered a nonlinear scalarization function related to free disposal sets. Then, by using this function, they established sufficient conditions for various types of semicontinuity such as B-semicontinuity, H-semicontinuity, and outer/inner continuity for solution maps of quasivariational inequalities involving free disposal sets. Then, in 2022, Shao et al. [52] investigated the connectedness of solution sets for generalized vector equilibrium problems involving free disposal sets within a complete metric space. Utilizing the Gerstewitz scalarization function and the oriented distance function, the researchers developed a novel scalarization approach and analyzed its properties. Utilizing this function, the authors investigated the existence of solutions for the scalarization problems thereby establishing a relationship between the solution sets of the reference problems and the corresponding scalarization problems. Additionally, Kiyani et al. [34] also used the oriented distance function and the non-convex separation function to establish optimality conditions for efficient solution of the unconstrained vector optimization problem. Recently, Tung and Duy [56] utilized the sequential compactness of feasible sets and the uniform coerciveness of the objective functions to examine the convergence in the sense of Painlevé-Kuratowski for efficient and approximate solutions of vector optimization problems involving the free disposal set. Very recently, in 2024, Zhou et al. [61] considered and investigated the properties of a generalized nonlinear Gerstewitz function for free disposal sets. They used these properties to study the stability in terms of lower and upper semicontinuities for solution maps of parametric vector equilibrium problems involving free disposal sets.

Uncertain optimization problems are an important branch of optimization theory and have recently garnered significant research interest from mathematicians. These problems arise from the reality that input data in practical scenarios are often imprecise, which may result from measurement errors, lack of information, or the inherent randomness of systems. It is also noteworthy that in practical situations, data containing imperfect information, as mentioned above, is almost unavoidable. Therefore, optimization problems with uncertain data play a crucial role in addressing complex challenges in the real world [19, 35, 50]. The literature highlights two primary approaches to addressing optimization problems with uncertain data: stochastic optimization and robust optimization. Stochastic optimization relies on prior knowledge of a specific probability distribution associated with the uncertain data. Its objective is to identify a solution that satisfies feasibility constraints with a given probability while maximizing the expected value of a cost function [42]. In contrast, robust optimization does not require any assumptions about probability distributions. Instead, it defines an uncertainty set for the data and aims to determine a solution that remains feasible under all possible scenarios within this set, a property referred to as robust feasibility [2, 15, 32, 47, 57]. For more details insights into the role and significance of robust optimization problems in meeting practical application demands, as well as the tools and techniques used for these problems, we would like to prefer readers to some important works on this topic [5, 36].

Motivated by the above observations, this paper aims to study robust uncertain optimization problems and examine the qualitative properties of their Benson efficient solutions. Specifically, in Sect. 2, we recall some fundamental concepts, including the semicontinuity of set-valued maps, convex separation theorems, and generalized convexlikeness properties of vector-valued maps. In Sect. 3, we derive the necessary and sufficient optimality conditions for such solutions of both unconstrained and constrained uncertain vector optimization problems through the use of the linear scalarization method. Subsequently, in Sect. 4, we investigate the semicontinuity properties of the robust Benson efficient solution maps for such problems, utilizing the scalarization representation of the solution sets. Following this, Sect. 5 introduces the concept of Hausdorff well-posedness for these problems, and by applying the linear scalarization method, we derive well-posedness conditions for the reference problems. Finally, some concluding remarks are presented in the last section, Sect. 6.

2 Preliminaries

Let $\mathbb{X}, \mathbb{Y}, \mathbb{W}$ be normed spaces and $\mathbb{Y}^*$ be the topological dual space of $\mathbb{Y}$, and K be a closed convex pointed cone in $\mathbb{Y}$ with nonempty interior. The family of all the nonempty subsets of $\mathbb{X}$ and $\mathbb{Y}$ are denoted by $\mathbf{P}(\mathbb{X})$ and $\mathbf{P}(\mathbb{Y})$, respectively. Let D be a nonempty subset of $\mathbb{Y}$, the topological interior and closure of D are denoted by $\operatorname{int} D$ and $\operatorname{cl} D$, respectively. The dual set and the strictly dual set of D are defined by

$$D^* := \{\tau \in \mathbb{Y}^* : \tau(d) \geq 0 \text{ for all } d \in D\},$$

and

$$D^\# := \{\tau \in \mathbb{Y}^* : \tau(d) > 0 \text{ for all } d \in D \setminus \{0_\mathbb{Y}\}\},$$

respectively. Obviously,

$$(-D)^* = -(D)^*.$$

Lemma 2.1 [62, Lemma 2.2] *If $D \subset \mathbb{Y}$ is a convex set with nonempty interior, then*

$$D^* = (\operatorname{int} D)^*.$$

Definition 2.1 [40, Definition 1.5] Let $B \in \mathbf{P}(\mathbb{Y})$. Then, we say that

(a) B generates the cone K, written as $K = \operatorname{cone}(B)$, if

$$K = \operatorname{cone}(B) = \{tb : t \geq 0, b \in B\};$$

(b) B is a base of K if B does not contain zero and for each $k \in K, k \neq 0_\mathbb{Y}$, there are unique $b \in B, t > 0$ such that $k = tb$.

The closure of $\operatorname{cone}(D)$ is denoted by $\operatorname{clcone}(D)$. Based on [31, Lemma 3.21], one has

$$\operatorname{int} K = \{y \in \mathbb{Y} : \tau(y) > 0 \text{ for all } \tau \in K^* \setminus \{0_{\mathbb{Y}^*}\}\}. \tag{1}$$

In the rest of this section, we consider $\Phi, \Phi_i : \mathbb{X} \rightrightarrows \mathbb{Y}$ being set-valued maps, $\zeta : \mathbb{X} \to \mathbb{Y}$ and $\eta : \mathbb{X} \to \mathbb{W}$ being vector-valued maps. We recall concepts of (semi)continuity and their properties.

Definition 2.2 [22, Definition 2.5.1] The map Φ is said to be

(a) *upper semicontinuous* (usc) at $x_0 \in \mathbb{X}$ if for any neighborhood Ξ of $\Phi(x_0)$, there exists a neighborhood Δ of x_0 such that $\Phi(x) \subseteq \Xi$ for all $x \in \Delta$;
(b) *Hausdorff upper semicontinuous* (Husc) at $x_0 \in \mathbb{X}$ if for each neighborhood Ξ of the origin in $\mathbb{Y}$, there exists a neighborhood Δ of x_0 such that $\Phi(x) \subseteq \Phi(x_0) + \Xi$ for all $x \in \Delta$;
(c) *lower semicontinuous* (lsc) at $x_0 \in \mathbb{X}$ if for any open subset Ξ of $\mathbb{Y}$ with $\Phi(x_0) \cap \Xi \neq \emptyset$, there exists a neighborhood Δ of x_0 such that $\Phi(x) \cap \Xi \neq \emptyset$ for all $x \in \Delta$;
(d) *Hausdorff lower semicontinuous* (Hlsc) at $x_0 \in \mathbb{X}$ if for each neighborhood Ξ of the origin in $\mathbb{Y}$, there exists a neighborhood Δ of x_0 such that $\Phi(x_0) \subseteq \Phi(x) + \Xi$ for all $x \in \Delta$.

It is said that Φ is usc, Husc, lsc, Hlsc on a nonempty subset Ω of $\mathbb{X}$ if Φ is usc, Husc, lsc, Hlsc at every element $x \in \Omega$, respectively. We also say that Φ is continuous, Hausdorff continuous on Ω if it is both usc and lsc, Husc and Hlsc on Ω, respectively.

Lemma 2.2 *The following statements are true*

(a) [44, Proposition 2.6] *Φ is lower semicontinuous at x_0 iff for any sequense $\{x_n\} \subset \mathbb{X}$ converging to x_0 and $y_0 \in \Phi(x_0)$, then there exists a sequence $\{y_n\}$ with $y_n \in \Phi(x_n)$ such that $\{y_n\}$ converges to y_0, or equivalently $\Phi(x_0) \subseteq \liminf \Phi(x_n) := \{y_0 \in \mathbb{Y} : \exists y_n \in \Phi(x_n), y_n \to y_0\}$.*

(b) [44, Proposition 2.19] *If $\Phi(x_0)$ is compact, then Φ is upper semicontinuous at x_0 iff for any sequence $\{x_n\} \subset \mathbb{X}$ converging to x_0 and $y_n \in \Phi(x_n)$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges to $y_0 \in \Phi(x_0)$.*

Lemma 2.3 [44, Theorem 2.68] *The following statements hold true.*

(a) *If Φ is usc at x_0, then Φ is Husc at x_0. Conversely, if Φ is Husc at x_0 and $\Phi(x_0)$ is compact, then Φ is usc at x_0.*

(b) *If Φ is Hlsc at x_0, then Φ is lsc at x_0. Conversely, if Φ is lsc at x_0 and $\Phi(x_0)$ is compact, then Φ is Hlsc at x_0.*

Lemma 2.4 [9, Theorem 2] *The union $\Phi = \bigcup_{i \in I} \Phi_i$ of a family of lsc set-valued maps Φ_i is also a lsc set-valued map, where I is an index set.*

Definition 2.3 [40, Definition 5.1] Let $x_0 \in \mathbb{X}$. The map η is said to be

(a) *K-lower semicontinuous* (K-lsc) at x_0 if for any neighborhood Ξ of $\eta(x_0)$, there exists a neighborhood Δ of x_0 such that $\eta(\Delta) \subseteq \Xi + K$;

(b) *K-upper semicontinuous* (K-usc) at x_0 if for any neighborhood Ξ of $\eta(x_0)$, there exists a neighborhood Δ of x_0 such that $\eta(\Delta) \subseteq \Xi - K$;

where $\eta(\Delta) = \{\eta(x) : x \in \Delta\}$.

Lemma 2.5 [27, Lemma 4.2] *Let $\{\varphi_n\} \subset \mathbb{Y}^*$ with $\varphi_n \xrightarrow{w^*} \varphi_0$, that is, $\varphi_n(z) \to \varphi_0(z)$ for every $z \in \mathbb{Y}$ and $\{z_n\} \subset \mathbb{Y}$ with $z_n \to z_0$. Then, $\varphi_n(z_n) \to \varphi_0(z_0)$.*

Now we recall results of the separation for convex sets, which are used in the next sections in this page.

Lemma 2.6 *Let $C_1, C_2 \in \mathbf{P}(\mathbb{Y})$.*

(a) [39, Lemma 2.1] *If C_1, C_2 are closed convex cones such that $C_1 \cap (-C_2) = \{0_{\mathbb{Y}}\}$, C_2 is pointed and has a compact base, then $C_2^{\#} \cap C_1^* \neq \emptyset$.*

(b) [53, Lemma 1.4] *If C_1, C_2 are cones in $\mathbb{Y}$ such that $C_1 \cap C_2 = \{0_{\mathbb{Y}}\}$, C_1 is closed and C_2 has a compact base, then there exists a pointed convex cone S such that $C_2 \backslash \{0_{\mathbb{Y}}\} \subset \operatorname{int} S$ and $C_1 \cap S = \{0_{\mathbb{Y}}\}$.*

(c) [31, Theorem 3.16] *If C_1, C_2 are convex with $\operatorname{int} C_1 \neq \emptyset$. Then, $\operatorname{int} C_1 \cap C_2 = \emptyset$ if and only if there exist $\tau \in \mathbb{Y}^* \backslash \{0_{\mathbb{Y}^*}\}$ and $r \in \mathbb{R}$ such that*

$$\tau(c_1) \leq r \leq \tau(c_2) \text{ for all } (c_1, c_2) \in C_1 \times C_2,$$

and

$$\tau(c) < r \text{ for all } c \in \operatorname{int} C_1.$$

Definition 2.4 Let $D \in \mathbf{P}(\mathbb{Y})$. Then, D is called

(a) [43, Definition 5] a *free disposal set* with respect to (w.r.t.) K if $D + K = D$;
(b) [12, Definition 2.5] an *improvement set* with respect to (w.r.t.) K if $0_{\mathbb{Y}} \notin D$ and $D + K = D$.

Remark 2.1 Based on the definitions, if D is an improvement set or a convex cone, then it is a free disposal set.

Inspired by [1, 51], the concepts of generalized convexlikeness for a vector-valued map are revisited through the following definition and lemma, which serve as key tools for the main results in the subsequent sections.

Definition 2.5 Let $\Omega \in \mathbf{P}(\mathbb{X})$ and $D \in \mathbf{P}(\mathbb{Y})$. Then, ζ is said to be

(a) *D-convexlike* on Ω if the set $\{\zeta(x) : x \in \Omega\} + D$ is convex;
(b) *D-subconvexlike* on Ω if the set $\{\zeta(x) : x \in \Omega\} + \operatorname{int} D$ is convex;
(c) *nearly D-subconvexlike* on Ω if the set $\operatorname{clcone}(\{\zeta(x) : x \in \Omega\} + D))$ is convex.

Lemma 2.7 *Let $\Omega \in \mathbf{P}(\mathbb{X})$, $D \in \mathbf{P}(\mathbb{Y})$, $E \in \mathbf{P}(\mathbb{W})$ and the map $(\zeta, \eta) : \Omega \to \mathbb{Y} \times \mathbb{W}$ defined by*

$$(\zeta, \eta)(x) := \zeta(x) \times \eta(x) \text{ for all } x \in \Omega.$$

If ζ is nearly D-subconvexlike on Ω and η is nearly E-subconvexlike on Ω, then (ζ, η) is nearly $(D \times E)$-subconvexlike on Ω.

Proof Let $(y_1, w_1), (y_2, w_2) \in \operatorname{clcone}(\{(\zeta(x), \eta(x)) : x \in \Omega\} + (D \times E))$ and $t \in [0, 1]$ be arbitrary. We have

$$y_1, y_2 \in \operatorname{clcone}(\{\zeta(x) : x \in \Omega\} + D) \quad \text{and} \quad w_1, w_2 \in \operatorname{clcone}(\{\eta(x) : x \in \Omega\} + E).$$

Then, there exist sequences $\{\lambda_n^i\}, \{\alpha_n^i\} \subset \mathbb{R}_+, \{u_n^i\} \subset \{\zeta(x) : x \in \Omega\}, \{d_n^i\} \subset D, \{z_n^i\} \subset \{\eta(x) : x \in \Omega\}$ and $\{v_n^i\} \subset E$ such that

$$\lambda_n^i(u_n^i + d_n^i) \to y_i \quad \text{and} \quad \alpha_n^i(z_n^i + v_n^i) \to w_i,$$

for all $i \in \{1, 2\}$. It leads to

$$t\lambda_n^1(u_n^1 + d_n^1) + (1-t)\lambda_n^2(u_n^2 + d_n^2) \to ty_1 + (1-t)y_2,$$

and

$$t\alpha_n^1(z_n^1 + v_n^1) + (1-t)\alpha_n^2(z_n^2 + v_n^2) \to tw_1 + (1-t)w_2.$$

Because ζ is nearly D-subconvexlike and η is nearly E-subconvexlike on Ω, $\operatorname{clcone}(\{\zeta(x) : x \in \Omega\} + D)$ and $\operatorname{clcone}(\{\eta(x) : x \in \Omega\} + E)$ are convex sets. Thus,

$$ty_1 + (1-t)y_2 \in \operatorname{clcone}(\{\zeta(x) : x \in \Omega\} + D),$$

and

$$tw_1 + (1-t)w_2 \in \operatorname{clcone}(\{\eta(x) : x \in \Omega\} + E).$$

Consequently,

$$(ty_1 + (1-t)y_2, tw_1 + (1-t)w_2) = t(y_1, w_1) + (1-t)(y_2, w_2)$$
$$\in \operatorname{clcone}\left(\{(\zeta(x), \eta(x)) : x \in \Omega\} + (D \times E)\right),$$

and hence the set $\operatorname{clcone}(\{(\zeta(x), \eta(x)) : x \in \Omega\} + (D \times E))$ is convex. Therefore, the map (ζ, η) is nearly $(D \times E)$-subconvexlike on Ω. $\square$

Definition 2.6 [55, Definition 2.1] Let $\Omega \in \mathbf{P}(\mathbb{X})$. The map η is said to be *naturally K-quasiconvex* on Ω if for all $x_1, x_2 \in \Omega$ and $\lambda \in]0, 1[$, there exists some $\mu \in]0, 1[$ such that

$$\eta(\lambda x_1 + (1-\lambda)x_2) \in \mu\eta(x_1) + (1-\mu)\eta(x_2) - K.$$

3 Optimality Conditions for Robust Benson Efficient Solutions

Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{W}$, K be defined as in Sect. 2, and $\mathbb{U}$ be a normed space. Let $\Omega \in \mathbf{P}(\mathbb{X})$ and $\zeta : \Omega \to \mathbb{Y}$ be a vector-valued map, we consider the following problem

$$(\text{VOP}) \quad \min_{x \in \Omega} \zeta(x).$$

Now, we consider a case of the objective map ζ also depends on parametric θ which is unknown or uncertain. Let $\Theta \subseteq \mathbb{U}$ be an uncertainty set reflecting the potential scenarios that may occur, and $\zeta : \Omega \times \Theta \to \mathbb{Y}$ be a map, we focus on the following uncertain problem

$$(\text{UVOP}) \quad \min_{\substack{x \in \Omega \\ \theta \in \Theta}} \zeta(x, \theta).$$

In what follows, Ω is called the feasible solution set and $\theta \in \Theta$ is called a scenario.

Let $D \in \mathbf{P}(\mathbb{Y})$ be a free disposal set w.r.t. K. Motivated by [56, 57], we propose the following concept of robust Benson efficient solution of (UVOP) involving the free disposal set D as follows.

Definition 3.1 Let $\theta_0 \in \Theta$ be given. An element $x_0 \in \Omega$ is called *a robust Benson efficient solution* of (UVOP) corresponding to the scenario θ_0, written as $x_0 \in \mathrm{BEff}_K(\text{UVOP})(\zeta, D)$, if

$$\operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D\right) \cap (-K) = \{0_{\mathbb{Y}}\}.$$

Remark 3.1 (i) The study of solutions for uncertain optimization problems is a significant topic and has attracted considerable interest from researchers. One common approach to these problems, used in many studies, is to address their solutions through appropriate set-valued optimization problems. By this way, until now, there are many interesting works on various solution properties for uncertain optimization problems, such as existence conditions [47], optimality conditions [11, 34, 37], and stability conditions [2, 14, 47]. Using this approach, researchers have discovered many valuable solution properties for these problems. However, a major drawback of this approach is that, for each given efficient solution, the concept of efficiency only indicates the existence of a scenario that illustrates efficiency of the solution without identifying this specific scenario, see e.g., [2, 11, 14, 34, 37, 47] and the references therein, and consequently these efficient solutions have limited applicability to practical situations. This is the key difference between the concept of efficient solution in Definition 3.1 and the existing concepts in [2, 11, 14, 34, 37, 47]. Specifically, the efficient solution concept proposed in Definition 3.1 is devoted to a given specific scenario, for instance the current scenario of the problem under consideration. The motivation of the idea behind this concept comes from many practical situations such as when we want to make decisions for some works related to the weather factors such as rain, sunshine, wind, or storms. In this case, factors that related to weather are uncertain scenarios, making decisions corresponds to a specific scenario deemed highly likely would be far more practical and, of course, would gain more support than considering a solution tied to an as-yet-undefined weather phenomenon. Similarly, in the case of making production decisions based on uncertain factors such as consumer trends or shopping habits, the concept of an efficient solution in Definition 3.1 can be viewed as a solution tied to a specific scenario, derived from surveys or data collected at the current time. Unfortunately, due to the complex structure, Benson efficient solutions have not yet been considered for set optimization problems based on set ordering relations. As a result, there are no any works to allow us to compare the Benson efficient solution defined in Definition 3.1 with solutions approached involving set optimization problems as mentioned above.

(ii) In order to explore relationships between the efficient solution in Definition 3.1 and concepts from related works, we examine two special cases of the free disposal set D as follows:
− When $D = K$, the efficient solution considered in Definition 3.1 coincides with the concept of solution in Definition 4.2 of [57], where the authors used the higher-order weak radial epiderivative to investigate optimality conditions for robust Benson efficient solutions with respect to the ordering cone K. More specifically, when $\Theta = \{\theta_0\}$, then Definition 3.1 corresponds to Definition 3.1 in [59].
− If D is a improvement set with respect to K and $\Theta = \{\theta_0\}$, then Definition 3.1 reduces to Definition 3.6 in [59], Definition 2.6 in [28], and Definition 2.9 in [27]. To the best of our knowledge, there have not been any works on robust Benson efficient solutions of uncertain vector optimization problems involving improvement sets.

The following example is provided to illustrate the computations for the efficient solution in Definition 3.1.

Example 3.1 Let $\mathbb{X} = \mathbb{Y} = \mathbb{U} = \mathbb{R}$, $K = \mathbb{R}_+$, $D = 1 + \mathbb{R}_+$, $\Omega = [0, 2]$ $\Theta = [0, 1]$, $\theta_0 \in \Theta$, and $\zeta : \Omega \times \Theta \to \mathbb{Y}$ defined by

$$\zeta(x, \theta) := x^2 + \theta^2 \text{ for all } (x, \theta) \in \Omega \times \Theta.$$

Then, $x_0 \in \Omega$ is a robust Benson efficient solution if and only if

$$\operatorname{clcone}\left(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Omega \times \Theta\} + D\right) \cap (-K) = \{0_{\mathbb{Y}}\},$$

or equivalently

$$\operatorname{clcone}\left(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Omega \times \Theta\} + D\right) \cap (-K \setminus \{0_{\mathbb{Y}}\}) = \emptyset. \tag{2}$$

On the other hand, we have

$$\zeta(x,\theta) - \zeta(x_0,\theta_0) + D = x^2 + \theta^2 - x_0^2 - \theta_0^2 + 1 + \mathbb{R}_+.$$

Therefore,

$$\begin{aligned}
\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Omega \times \Theta\} + D &= [-x_0^2 - \theta_0^2, -x_0^2 - \theta_0^2 + 5] + 1 + \mathbb{R}_+ \\
&= [-x_0^2 - \theta_0^2 + 1, -x_0^2 - \theta_0^2 + 6] + \mathbb{R}_+ \\
&= [-x_0^2 - \theta_0^2 + 1, +\infty[.
\end{aligned}$$

Hence, statement (2) is satisfied if and only if $x_0^2 \leq 1 - \theta_0^2$, and consequently

$$\operatorname{BEff}_K(\text{UVOP})(\zeta, D) = \left[0, \sqrt{1-\theta_0^2}\right].$$

For convenience in presentation, we consider the following assumptions, which will be employed in the subsequent discussions.

($\mathcal{A}0$) K has a compact base.
($\mathcal{A}1$) ζ is nearly D-subconvexlike on $\Omega \times \Theta$.

The following result establishes sufficient optimality conditions for the robust Benson efficient solution of the problem (UVOP) corresponding to the scenario θ_0.

Theorem 3.1 *Let $\tau \in K^{\#}$ and $\theta_0 \in \Theta$ be given. Assume that $x_0 \in \Omega$ satisfying*

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left(\tau(\zeta(x_0,\theta_0)) - \tau(\zeta(x,\theta))\right) \leq \inf_{d \in D} \tau(d). \tag{3}$$

Then, $x_0 \in \operatorname{BEff}_K(\text{UVOP})(\zeta, D)$.

Proof It is clear that $0_{\mathbb{Y}} \in \operatorname{clcone}\left(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Omega \times \Theta\} + D\right) \cap (-K)$. Let $a \in \operatorname{clcone}\left(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Omega \times \Theta\} + D\right) \cap (-K)$ be arbitrary, we will show that $a = 0_{\mathbb{Y}}$. Because D is a free disposal set w.r.t. K that

$$a \in \operatorname{clcone}\left(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Omega \times \Theta\} + D + K\right) \cap (-K).$$

Since $a \in \operatorname{clcone}(\{\zeta(x,\theta)-\zeta(x_0,\theta_0):(x,\theta)\in\Omega\times\Theta\}+D+K)$, there are $\{t_n\}\subset\mathbb{R}_+$, $\{(x_n,\theta_n)\}\subset\Omega\times\Theta$, $\{d_n\}\subset D$ and $\{v_n\}\subset K$ such that

$$t_n(\zeta(x_n,\theta_n)-\zeta(x_0,\theta_0)+d_n+v_n)\to a.$$

In view of $\tau\in K^\#$, we obtain

$$t_n\,(\tau(\zeta(x_n,\theta_n))-\tau(\zeta(x_0,\theta_0))+\tau(d_n)+\tau(v_n))\to\tau(a). \tag{4}$$

Since inequality (3) holds, we have

$$\tau(\zeta(x_0,\theta_0))-\tau(\zeta(x_n,\theta_n))\leq\tau(d_n),$$

which implies that

$$\tau(\zeta(x_n,\theta_n))-\tau(\zeta(x_0,\theta_0))+\tau(d_n)\geq 0. \tag{5}$$

On the other hand, it follows from $\tau\in K^\#$ and $v_n\in K$ that

$$\tau(v_n)\geq 0. \tag{6}$$

By employing (4), (5) and (6), we obtain

$$\tau(a)\geq 0. \tag{7}$$

Furthermore, since $\tau\in K^\#$ and $a\in -K$, we have $\tau(a)\leq 0$. This together with (7) implies that $\tau(a)=0$. Consequently, $a=0_{\mathbb{Y}}$ as $\tau\in K^\#$, and thus

$$\operatorname{clcone}(\{\zeta(x,\theta)-\zeta(x_0,\theta_0):(x,\theta)\in\Omega\times\Theta\}+D)\cap(-K)=\{0_{\mathbb{Y}}\}.$$

Therefore, $x_0\in\operatorname{BEff}_K\,(\text{UVOP})(\zeta,D)$. □

The following example is provided to illustrate the applicability of Theorem 3.1.

Example 3.2 Let $\mathbb{X}=\mathbb{R}$, $\mathbb{Y}=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, $D=(1,1)+\mathbb{R}^2_+$, $\Omega=[0,2]$, $\Theta=[-1,1]$ and $\theta_0=\frac{1}{2}$. The vector-valued map $\zeta:\Omega\times\Theta\to\mathbb{Y}$ is defined by

$$\zeta(x,\theta):=(x+|\theta|,x^4+\theta^4)\ \text{ for all } x\in\Omega.$$

Let $x_0=0\in[0,2]$. We have

$$\zeta\left(0,\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{16}\right).$$

Taking $\tau = (1, 1)$, we obtain

$$\sup_{\substack{x\in\Omega\\ \theta\in\Theta}} \left\{ \frac{9}{16} - x - |\theta| - x^4 - \theta^4 \right\} = \frac{9}{16} \leq \inf_{(d_1,d_2)\in D} \{d_1 + d_2\} = 2,$$

which provides that

$$\sup_{\substack{x\in\Omega\\ \theta\in\Theta}} \left\{ \tau \circ \zeta \left(0, \frac{1}{2}\right) - \tau \circ \zeta(x, \theta) \right\} \leq \inf_{d\in D} \tau(d).$$

By Theorem 3.1, we obtain $x_0 = 0 \in \mathrm{BEff}_K\,(\mathrm{UVOP})(\zeta, D)$.

Now, we employ the compact base property of K together with the subconvexlikeness of ζ to consider necessary optimality conditions for robust Benson efficient solutions of (UVOP) corresponding to the scenario θ_0.

Theorem 3.2 *Let $\theta_0 \in \Theta$ be given. Assume that assumptions $(\mathcal{A}0)$ and $(\mathcal{A}1)$ are satisfied and $x_0 \in \mathrm{BEff}_K\,(\mathrm{UVOP})(\zeta, D)$, then there exists $\tau \in K^\#$ such that x_0 satisfies inequality (3).*

Proof It follows from $x_0 \in \mathrm{BEff}_K(\mathrm{UVOP})(\zeta, D)$ that

$$\operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D\right) \cap (-K) = \{0_{\mathbb{Y}}\}. \tag{8}$$

Because ζ is nearly D-subconvexlike on $\Omega \times \Theta$, we obtain that

$$\operatorname{clcone}\left(\{\zeta(x, \theta) : (x, \theta) \in \Omega \times \Theta\} + D\right)$$

is convex, which implies that the set $\operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D\right)$ is also convex. Combining this with (8) and assumption $(\mathcal{A}0)$, Lemma 2.6(a) yields that there exists $\hat{\tau} \in K^\#$ satisfying

$$\hat{\tau} \in \left(\operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D\right)\right)^*,$$

and consequently

$$\hat{\tau}(z) \geq 0 \text{ for all } z \in \operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D\right).$$

Since $\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D$ is a subset of the set

$$\operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D\right),$$

we also have

$$\hat{\tau}(z) \geq 0 \text{ for all } z \in \{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Omega \times \Theta\} + D.$$

Therefore,

$$\hat{\tau}(\zeta(x,\theta) - \zeta(x_0,\theta_0) + d) \geq 0 \text{ for all } (x,\theta,d) \in \Omega \times \Theta \times D,$$

which leads to

$$\hat{\tau}(\zeta(x_0,\theta_0)) - \hat{\tau}(\zeta(x,\theta)) \leq \hat{\tau}(d) \text{ for all } (x,\theta,d) \in \Omega \times \Theta \times D.$$

Consequently,

$$\sup_{\substack{x\in\Omega \\ \theta\in\Theta}} (\tau(\zeta(x_0,\theta_0)) - \tau(\zeta(x,\theta))) \leq \inf_{d\in D} \tau(d).$$

In other words, inequality (3) is satisfied. □

We now consider a very important case where $\mathbb{Y} = \mathbb{R}^n$, namely when the problem (UVOP) becomes a multi-objective optimization problem. By Remark 1.6 in [40], the closed pointed convex cone K has a compact base, and hence assumption ($\mathcal{A}0$) is satisfied. The following result is obtained from Theorem 3.2.

Corollary 3.1 *Let $\theta_0 \in \Theta$. Assume that assumption ($\mathcal{A}1$) holds and $x_0 \in \mathrm{BEff}_K(\mathrm{UVOP})(\zeta, D)$, then there exists $\tau \in K^\#$ such that x_0 satisfies inequality (3).*

Turning to the constrained uncertain vector optimization problems, let $C \subset \mathbb{W}$ be a closed pointed convex cone with nonempty interior. We denote by $\eta : \Omega \times \Theta \to \mathbb{W}$ a vector-valued map and consider the following constrained uncertain vector optimization problem:

$$(\mathrm{CUVOP}) \qquad \min_{\substack{x\in\Omega,\theta\in\Theta \\ \eta(x,\theta)\in -C}} \zeta(x,\theta).$$

We will define a concept of robust Benson efficient solution of the problem (CUVOP) as follows.

Definition 3.2 Let $\theta_0 \in \Theta$ be given. The element $x_0 \in \Omega$ is termed *a robust Benson efficient solution* of (CUVOP) corresponding to θ_0, written as $x_0 \in \mathrm{BEff}_K(\mathrm{CUVOP})(\zeta, \eta, D)$, if

$$\mathrm{clcone}\,(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : \eta(x,\theta) \in -C, (x,\theta) \in \Omega \times \Theta\} + D) \cap (-K) = \{0_{\mathbb{Y}}\}.$$

Setting $\Sigma := \{(x,\theta) \in \Omega \times \Theta : \eta(x,\theta) \in -C\}$, by the same techniques as in the proof of Theorem 3.1, we also obtain sufficient conditions for robust Benson efficient solutions of the problem (CUVOP) corresponding to the scenario θ_0 presented in the following theorem.

Theorem 3.3 *Let $\theta_0 \in \Theta$ be given. If $x_0 \in \Omega$ and there exists $(\tau, \iota) \in K^\# \times C^*$ such that*

$$\sup_{\substack{x\in\Omega \\ \theta\in\Theta}} [\tau(\zeta(x_0,\theta_0)) - (\tau(\zeta(x,\theta)) + \iota(\eta(x,\theta)))] \leq \inf_{d\in D} \tau(d), \tag{9}$$

then $x_0 \in \mathrm{BEff}_K(\mathrm{CUVOP})(\zeta, \eta, D)$.

An example is presented below to showcase the applicability of Theorem 3.3.

Example 3.3 Let $\mathbb{X} = \mathbb{U} = \mathbb{R}$, $\mathbb{Y} = \mathbb{W} = \mathbb{R}^2$, $K = C = \mathbb{R}^2_+$, $\Omega = [-2, 2]$, $\Theta = [0, 2]$, $\theta_0 = 0$, and $D = \{(x, y) \in \mathbb{R}^2 : y \geq 3 - x\} \cap \{(x, y) \in \mathbb{R}^2 : y \geq 3\}$. The vector-valued maps $\zeta : \Omega \times \Theta \to \mathbb{Y}$ and $\eta : \Omega \times \Theta \to \mathbb{W}$ are, respectively, defined by

$$\zeta(x, \theta) := 2^{-\theta} \left(\frac{1}{3}x, \frac{1}{4}x^2 \right) \text{ for all } (x, \theta) \in \Omega \times \Theta,$$

and

$$\eta(x, \theta) := (x - \theta, -\theta) \text{ for all } (x, \theta) \in \Omega \times \Theta.$$

It is obvious that $\Sigma := \{(x, \theta) \in \Omega \times \Theta : x \leq \theta, 0 \leq \theta \leq 2\}$. Choosing $x_0 = 1 \in \Omega$, we have $\zeta(1, 0) = \left(\frac{1}{3}, \frac{1}{4}\right)$. For $\tau = (1, 1)$ and $\iota = (0, 1)$, by direct computations, we get

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \{((1, 1)(\zeta(1, 0))) - ((1, 1)(\zeta(x, \theta)) + (0, 1)(\eta(x, \theta)))\} =$$
$$= \sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left\{ \frac{7}{12} - \left(2^{-\theta} \left(\frac{1}{3}x + \frac{1}{4}x^2 \right) - \theta \right) \right\} = \frac{94}{36},$$

and

$$\inf_{(d_1, d_2) \in D} (1, 1)(d_1, d_2) = 3,$$

which will imply that

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} [(1, 1)(\zeta(1, 0)) - ((1, 1)(\zeta(x, \theta)) + (0, 1)(\eta(x, \theta)))] \leq \inf_{d \in D} (1, 1)(d).$$

Consequently, inequality (9) is satisfied. By applying Theorem 3.3, we conclude that $x_0 \in \mathrm{BEff}_K(\mathrm{CUVOP})(\zeta, \eta, D)$.

Motivated by [39, Definition 2.2], we consider the following assumption to discuss necessary optimality conditions for robust Benson efficient solutions.

($\mathcal{A}2$) (A generalized Slater Constraint Qualification condition) there exist $x \in \Omega$ and $\theta \in \Theta$ such that

$$\eta(x, \theta) \in -\operatorname{int} C.$$

We now discuss necessary optimality conditions for robust Benson efficient solutions of the problem (CUVOP) corresponding to the scenario θ_0 provided in the following result.

Theorem 3.4 *Let $\theta_0 \in \Theta$ be given. Assume that $x_0 \in \mathrm{BEff}_K\,(\mathrm{CUVOP})(\zeta, \eta, D)$ and assumptions ($\mathcal{A}0$)-($\mathcal{A}2$) are satisfied. Then, there exists $(\tau, \iota) \in K^{\#} \times C^*$ such that inequality (9) holds true.*

Proof First, since the problem (CUVOP) satisfies assumption ($\mathcal{A}2$), then it is clear that the set clcone $(\{\eta(x,\theta) : (x,\theta) \in \Omega \times \Theta\} + C)$ is convex, and hence η is nearly C-subconconvexlike on $\Omega \times \Theta$. Then, by using assumption ($\mathcal{A}1$) and Lemma 2.7, we imply that (ζ, η) is also nearly $(D \times C)$-subconvexlike on $\Omega \times \Theta$. Thus, clcone$(\{(\zeta(x,\theta), \eta(x,\theta)) : (x,\theta) \in \Omega \times \Theta\} + (D \times C))$ is convex, and so clcone $(\{(\zeta(x,\theta) - \zeta(x_0,\theta_0), \eta(x,\theta)) : (x,\theta) \in \Omega \times \Theta\} + (D \times C))$ is also a convex subset of $\mathbb{Y} \times \mathbb{W}$.

Next, it follows from $x_0 \in \mathrm{BEff}_K$ (CUVOP)(ζ, η, D) that

$$- (\mathrm{clcone}\,(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Sigma\} + D)) \cap K = \{0_{\mathbb{Y}}\}. \tag{10}$$

Setting $C_1 = -\,\mathrm{clcone}\,(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Sigma\} + D)$ and $C_2 = K$, we have $C_1 \cap C_2 = \{0_{\mathbb{Y}}\}$. Moreover, since C_1, C_2 are cones, C_1 is closed, and C_2 has a compact base, Lemma 2.6(b) implies that there exists a pointed convex cone $\widehat{S}$ such that

$$(K \setminus \{0_{\mathbb{Y}}\}) \subset \mathrm{int}\,\widehat{S},$$

and

$$-\,\mathrm{clcone}\,(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Sigma\} + D) \cap \widehat{S} = \{0_{\mathbb{Y}}\}.$$

It leads to

$$\mathrm{clcone}\,(\{\zeta(x,\theta) - \zeta(x_0,\theta_0) : (x,\theta) \in \Sigma\} + D) \cap (-\,\mathrm{int}\,\widehat{S}) = \emptyset. \tag{11}$$

We now show that

$$P \cap \big((-\,\mathrm{int}\,\widehat{S}) \times (-\,\mathrm{int}\,C)\big) = \emptyset, \tag{12}$$

where

$$P = \mathrm{clcone}\,(\{(\zeta(x,\theta) - \zeta(x_0,\theta_0), \eta(x,\theta)) : (x,\theta) \in \Omega \times \Theta\} + (D \times C)).$$

If (12) is not satisfied, we can find $(\hat{y}, \hat{w}) \in \mathbb{Y} \times \mathbb{W}$ such that

$$(\hat{y}, \hat{w}) \in P \cap \big((-\,\mathrm{int}\,\widehat{S}) \times (-\,\mathrm{int}\,C)\big). \tag{13}$$

Then, there are sequences $\{(d_n, c_n)\} \subset D \times C$, $\{\lambda_n\} \subset \mathbb{R}_+$, $\{x_n\} \subset \Omega$, and $\{\theta_n\} \subset \Theta$ such that

$$\lambda_n(\zeta(x_n,\theta_n) - \zeta(x_0,\theta_0) + d_n) \to \hat{y} \quad \text{and} \quad \lambda_n(\eta(x_n,\theta_n) + c_n) \to \hat{w}. \tag{14}$$

It is obvious that the sequence $\{\lambda_n(\eta(x_n,\theta_n) + c_n)\}$ converges to $\hat{w} \in -\,\mathrm{int}\,C$, and so there exists $\hat{n} \in \mathbb{N}$ such that

$$\lambda_n(\eta(x_n,\theta_n) + c_n) \in -\,\mathrm{int}\,C \ \text{ for all } n \geq \hat{n}.$$

On the other hand, because C is a cone and $\lambda_n \neq 0$, we have

$$\eta(x_n,\theta_n) + c_n \in -C \ \text{ for all } n \geq \hat{n}.$$

Consequently, $\eta(x_n, \theta_n) \in -C$, which implies that $(x_n, \theta_n) \in \Sigma$ for all $n \geq \hat{n}$. This together with (14) provides that

$$\hat{y} \in \operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Sigma\} + D\right).$$

Furthermore, in view of (13), we have $\hat{y} \in -\operatorname{int} \widehat{S}$, and hence

$$\hat{y} \in \operatorname{clcone}\left(\{\zeta(x, \theta) - \zeta(x_0, \theta_0) : (x, \theta) \in \Sigma\} + D\right) \cap (-\operatorname{int} \widehat{S}),$$

which is impossible due to (11). Thus, (12) holds true.

By Lemma 2.6(c), there exists $(\tau, \iota) \in (\mathbb{Y}^* \times \mathbb{W}^*) \setminus \{(0_{\mathbb{Y}^*}, 0_{\mathbb{W}^*})\}$ such that

$$(\tau, \iota)(y, w) \geq \tau(\hat{k}) + \iota(c),$$

for all $(y, w) \in \operatorname{clcone}\left(\{(\zeta(x, \theta) - \zeta(x_0, \theta_0), \eta(x, \theta)) : (x, \theta) \in \Omega \times \Theta\} + (D \times C)\right)$, and $(\hat{k}, c) \in (-\operatorname{int} \widehat{S}) \times (-\operatorname{int} C)$. Therefore,

$$\lambda(\tau(\zeta(x, \theta) - \zeta(x_0, \theta_0) + d) + \iota(\eta(x, \theta) + c_1)) \geq \tau(-\hat{k}_1) + \iota(-c_2), \tag{15}$$

for all $\lambda \geq 0$, $(x, \theta) \in \Omega \times \Theta$, $(d, c_1) \in D \times C$, and $(\hat{k}_1, c_2) \in \operatorname{int} \widehat{S} \times \operatorname{int} C$. For $c_1 = 0_{\mathbb{W}}$ and $\lambda > 0$, by (15), we obtain

$$\tau(\zeta(x, \theta) - \zeta(x_0, \theta_0) + d) + \iota(\eta(x, \theta)) \geq \frac{1}{\lambda}\Big(\tau(-\hat{k}_1) + \iota(-c_2)\Big),$$

for all $(x, \theta) \in \Omega \times \Theta$, $d \in D$ and $(\hat{k}_1, c_2) \in \operatorname{int} \widehat{S} \times \operatorname{int} C$. Let $\lambda \to \infty$, we get that

$$\tau(\zeta(x_0, \theta_0)) - (\tau(\zeta(x, \theta)) + \iota(\eta(x, \theta))) \leq \tau(d) \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D, \tag{16}$$

which implies that

$$\tau(\zeta(x_0, \theta_0)) - (\tau(\zeta(x, \theta)) + \iota(\eta(x, \theta))) \leq \inf_{d \in D} \tau(d) \text{ for all } (x, \theta) \in \Omega \times \Theta. \tag{17}$$

Consequently,

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left[\tau(\zeta(x_0, \theta_0)) - (\tau(\zeta(x, \theta)) + \iota(\eta(x, \theta)))\right] \leq \inf_{d \in D} \tau(d).$$

By (15), and choosing $\lambda = 0$, we have

$$\tau(\hat{k}_1) + \iota(c_2) \geq 0 \text{ for all } (\hat{k}_1, c_2) \in \operatorname{int} \widehat{S} \times \operatorname{int} C. \tag{18}$$

Now, we prove that $(\tau, \iota) \in K^{\#} \times C^*$. First, we suppose that there is $c_3 \in \operatorname{int} C$ such that $\iota(c_3) < 0$, then by (18), we have

$$\tau(\hat{k}) \geq -\iota(c_3) > 0 \text{ for all } \hat{k} \in \operatorname{int} \widehat{S}.$$

This implies that $-\frac{\tau(\hat{k})}{\iota(c_3)} \geq 1$, and hence

$$0 > \iota(c_3) = \iota\left(\left(-\frac{\tau(\hat{k})}{\iota(c_3)} + 1\right) c_3\right) + \tau(\hat{k}),$$

which contradicts (18) as $\left(-\frac{\tau(\hat{k})}{\iota(c_3)} + 1\right) \in \operatorname{int} C$. Therefore, we get that $\iota(c_2) \geq 0$ for all $c_2 \in \operatorname{int} C$.

Using the above techniques, we also obtain that $\tau(\hat{k}_1) \geq 0$ for all $\hat{k}_1 \in \operatorname{int} \widehat{S}$. Due to this and the fact that $\iota(c_2) \geq 0$ for all $c_2 \in \operatorname{int} C$, Lemma 2.1 leads to

$$(\tau, \iota) \in (\operatorname{int} \widehat{S})^* \times (\operatorname{int} C)^* = (\widehat{S})^* \times C^*.$$

If $\tau = 0_{(\widehat{S})^*}$, then by (16) and $\iota \neq 0_{C^*}$, one has

$$\iota(\eta(x, \theta)) \geq 0 \text{ for all } (x, \theta) \in \Omega \times \Theta. \tag{19}$$

Moreover, it follows from assumption $(\mathcal{A}2)$ that there exist $x_1 \in \Omega$ and $\theta_1 \in \Theta$ such that $\eta(x_1, \theta_1) \in (-\operatorname{int} C)$. Combining this with (1) and $\iota \in C^* \backslash \{0_{C^*}\}$, we have $\iota(\eta(x_1, \theta_1)) < 0$, which contradicts (19). Therefore, we conclude that $\tau \neq 0_{(\widehat{S})^*}$.

Let $\hat{d} \in K \setminus \{0_{\mathbb{Y}}\} \subset \operatorname{int} \widehat{S}$ be arbitrary. Then, there exists a balanced neighborhood Ξ of $0_{\mathbb{Y}}$ such that

$$\hat{d} + \Xi \subset \widehat{S}. \tag{20}$$

Sine $\tau \neq 0_{(\widehat{S})^*}$, there exists $\hat{v} \in \Xi$ such that $\tau(\hat{v}) > 0$, if not we have $\tau(y) = 0$ for all $y \in \mathbb{Y}$ which is a contradiction. Furthermore, since Ξ is a balanced neighborhood, we obtain $-\hat{v} \in \Xi$. It follows from (20) that $\hat{d} - \hat{v} \in \widehat{S}$, and thus

$$\tau(\hat{d}) - \tau(\hat{v}) = \tau(\hat{d} - \hat{v}) \geq 0.$$

This together with $\tau(\hat{v}) > 0$ implies that $\tau(\hat{d}) \geq \tau(\hat{v}) > 0$, and consequently $\tau \in K^\#$ as $\hat{d} \in K \setminus \{0_{\mathbb{Y}}\}$ is arbitrary. Therefore, $(\tau, \iota) \in K^\# \times C^*$, and hence the proof is completed. □

The next example is presented to demonstrate how Theorem 3.4 can be applied.

Example 3.4 Let $\mathbb{X} = \mathbb{U} = \mathbb{R}$, $\mathbb{Y} = \mathbb{W} = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $C = \mathbb{R}^2_+$, $\Omega = [-1, 2]$, $\Theta = [0, 2]$, $\theta_0 = 0$ and $D = (1, \frac{1}{2}) + \mathbb{R}^2_+$. The vector-valued maps $\zeta : \Omega \times \Theta \to \mathbb{Y}$ and $\eta : \Omega \times \Theta \to \mathbb{W}$ are, respectively, defined by

$$\zeta(x, \theta) := e^{\theta}(x, x^2) \quad \text{for all } (x, \theta) \in \Omega \times \Theta,$$

and

$$\eta(x, \theta) := (-x, \theta - x) \text{ for all } (x, \theta) \in \Omega \times \Theta.$$

It is not hard to check that

$$\Sigma = \{(x, \theta) \in \Omega \times \Theta : 0 \leq x \leq 2, x \geq \theta\}.$$

For $x_0 = 0$, we have $\zeta(0, 0) = (0, 0)$, and hence

$$\operatorname{clcone}\left(\{\zeta(x, \theta) - (0, 0) : (x, \theta) \in \Sigma\} + D\right) \cap (-K) = \{0_{\mathbb{R}^2}\}.$$

Therefore, $x_0 = 0 \in \mathrm{BEff}_K\,(\mathrm{CUVOP})(\zeta, \eta, D)$.

Moreover, assumption $(\mathcal{A}0)$ of Theorem 3.4 holds true as K has a compact base given by

$$B = \{(b_1, b_2) \in \mathbb{R}^2 : b_1 + b_2 = 1, 0 \le b_1 \le 1\}.$$

On the other hand, for $\hat{x} = 2$, $\hat{\theta} = 1$, we have

$$\eta(2, 1) = (-2, -1) \in -\operatorname{int} C,$$

and so assumption $(\mathcal{A}2)$ of Theorem 3.4 is satisfied.

It is also obvious that $\operatorname{clcone}\left(\{\zeta(x, \theta) : (x, \theta) \in \Omega \times \Theta\} + D\right) = \mathbb{R}^2_+$ is convex, and hence assumption $(\mathcal{A}1)$ of Theorem 3.4 holds.

By employing Theorem 3.4, there exists $(\tau, \iota) \in K^{\#} \times C^*$ such that

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left[\tau(\zeta(x_0, \theta_0)) - (\tau(\zeta(x, \theta)) + \iota(\eta(x, \theta)))\right] \le \inf_{d \in D} \tau(d). \tag{21}$$

In fact, taking $\tau = (1, 1)$, $\iota = (1, 0)$, we have

$$\begin{aligned}\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left[(1, 1)(0, 0) - ((1, 1)(\zeta(x, \theta)) + (1, 0)(\eta(x, \theta)))\right] &= \sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left[-(e^{\theta} x + e^{\theta} x^2 - x)\right] \\ &= \frac{(e^2 - 1)^2}{4e^2} \approx 1.38.\end{aligned}$$

Moreover,

$$\inf_{d \in D} (1, 1)(d) = \inf_{(d_1, d_2) \in D} (d_1 + d_2) = \frac{3}{2},$$

and hence (21) is satisfied.

We will finalize this section with applying obtained results to a very important special case of (CUVOP), namely when $\mathbb{Y}$ is a finite dimensional space. By applying Theorem 3.4, we obtain the following result.

Corollary 3.2 *Let $\mathbb{Y}$ be a finite dimensional space and $\theta_0 \in \Theta$ be given. Assume that $x_0 \in \mathrm{BEff}_K\,(\mathrm{CUVOP})(\zeta, \eta, D)$ and assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ are satisfied. Then, there exists $(\tau, \iota) \in K^{\#} \times C^*$ such that inequality (9) holds.*

Remark 3.2 Studying optimality conditions for efficient solutions of uncertain vector optimization problems is challenging due to the computational rules for generalized differentiation and subdifferentials of objective functions and constraint sets [13, 18, 48]. These rules become even more complex when we use to examine the optimality conditions for robust Benson efficient solutions of uncertain vector optimization problems. For this reason, up to now, we have not found any other works investigating optimality conditions for Benson efficient solutions of such problems, aside from Wang et al. [57], in which the authors studied optimality conditions for robust Benson efficient solutions involving ordering cones by using the higher-order weak radial epiderivative. Motivated by the above observations, this study does not pursue the research direction of using tools such as generalized differentiation and subdifferentials tailored to robust Benson efficient solutions in uncertain vector optimization problems. Instead, we approach the problem using scalarization methods. With this approach, we not only establish the optimality conditions presented in this section but also derive stability conditions which will be discussed in the following sections.

4 Semicontinuities Conditions for Robust Benson Efficient Solution Maps

Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{W}$, $\mathbb{U}$, K, C, Ω, Θ, D be defined as in Sect. 3, $\mathbb{P}$ be a normed space and $\Gamma \in \mathbf{P}(\mathbb{P})$ and let $\zeta : \Omega \times \Theta \times \Gamma \to \mathbb{Y}$ be a map. For each $\gamma \in \Gamma$, we aim to discuss qualitative properties of solutions of the following problem

$$\text{(PUVOP)} \quad \min_{\substack{x \in \Omega \\ \theta \in \Theta}} \quad \zeta(x, \theta, \gamma).$$

Let $\theta_0 \in \Theta$ be given. Motivated by Definition 3.1, we propose the concept of robust Benson efficient solution of (PUVOP) involving the free disposal set D as follows.

Definition 4.1 For each $\gamma \in \Gamma$, an element $x_0 \in \Omega$ is called *a robust Benson efficient solution* of (PUVOP) corresponding to the scenario θ_0, written as $x_0 \in \mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma)$, if

$$\operatorname{clcone}\left(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : (x, \theta) \in \Omega \times \Theta\} + D\right) \cap (-K) = \{0_{\mathbb{Y}}\}.$$

We consider the map $E : K^* \times \Gamma \rightrightarrows X$ defined by

$$(\tau, \gamma) \mapsto E(\tau, \gamma) := \left\{ x_0 \in \Omega : \sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left(\tau(\zeta(x_0, \theta_0, \gamma)) - \tau(\zeta(x, \theta, \gamma))\right) \le \inf_{d \in D} \tau(d) \right\},$$

for all $(\tau, \gamma) \in K^* \times \Gamma$, and we use it to study the scalar representation for the solution sets. Similar to the previous section, we also impose some assumptions to simplify the presentation.

($\mathcal{A}'1$) For each $\gamma \in \Gamma$, $\zeta(\cdot, \cdot, \gamma)$ is nearly D-subconvexlike on $\Omega \times \Theta$.
($\mathcal{A}3$) Ω is compact.
($\mathcal{A}'3$) Ω is compact and convex, Θ is compact.
($\mathcal{A}4$) ζ is continuous on $\Omega \times \Theta \times \Gamma$.
($\mathcal{A}5$) For each $\gamma \in \Gamma$, $\zeta(\cdot, \theta_0, \gamma)$ is naturally K-quasiconvex on Ω.

($\mathcal{A}6$) η is C-lower semicontinuous and naturally C-quasiconvex on $\Omega \times \Theta$.

In view of Theorems 3.1 and 3.2, we have the following scalar representation of robust Benson efficient solution set of (PUVOP) corresponding to the scenario θ_0.

Lemma 4.1 *For each $\gamma \in \Gamma$, we have*

$$E(\tau, \gamma) \subseteq \mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma),$$

for all $\tau \in K^{\#}$. Moreover, if assumptions ($\mathcal{A}0$) and ($\mathcal{A}'1$) are true, then

$$\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma) = \bigcup_{\tau \in K^{\#}} E(\tau, \gamma).$$

Now, we turn to studying the properties of the map E, and then we will use these properties, along with the results on the scalar representation obtained above, to examine the stability for the robust Benson efficient solutions of (PUVOP).

Theorem 4.1 *Assume that assumptions ($\mathcal{A}3$) and ($\mathcal{A}4$) are fulfilled. Then, E is compact valued and upper semicontinuous on $K^* \times \Gamma$.*

Proof Assumption ($\mathcal{A}3$) ensures the compact-valuedness of E by establishing its closed-valuedness. For every $(\tau_0, \gamma_0) \in K^* \times \Gamma$ and $\{x_n\} \subset E(\tau_0, \gamma_0)$ converging to x_0, we will show that $x_0 \in E(\tau_0, \gamma_0)$. Because $x_n \in \Omega$ and Ω is closed, we conclude that $x_0 \in \Omega$. Since $x_n \in E(\tau_0, \gamma_0)$, we have

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} (\tau_0(\zeta(x_n, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0))) \leq \inf_{d \in D} \tau_0(d),$$

which implies that

$$\tau_0(\zeta(x_n, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0)) \leq \inf_{d \in D} \tau_0(d) \text{ for all } (x, \theta) \in \Omega \times \Theta.$$

Combining this with the continuity of τ_0 and ζ, we obtain

$$\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0)) \leq \inf_{d \in D} \tau_0(d) \text{ for all } (x, \theta) \in \Omega \times \Theta.$$

Consequently,

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} (\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0))) \leq \inf_{d \in D} \tau_0(d),$$

namely $x_0 \in E(\tau_0, \gamma_0)$. Therefore, $E(\tau_0, \gamma_0)$ is a closed set, and hence it is a compact set.

Let $\{(\tau_n, \gamma_n)\} \subset K^* \times \Gamma$ converging to (τ_0, γ_0), and $x_n \in E(\tau_n, \gamma_n)$. It follows from $x_n \in E(\tau_n, \gamma_n)$ that

$$\sup_{\substack{x\in\Omega\\ \theta\in\Theta}} (\tau_n(\zeta(x_n, \theta_0, \gamma_n)) - \tau_n(\zeta(x, \theta, \gamma_n))) \leq \inf_{d\in D} \tau_n(d),$$

which implies that

$$\tau_n(\zeta(x_n, \theta_0, \gamma_n)) - \tau_n(\zeta(x, \theta, \gamma_n)) \leq \tau_n(d) \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D. \quad (22)$$

Because $x_n \in \Omega$ and Ω is compact, we can assume that $\{x_n\}$ converges to $x_0 \in \Omega$. Moreover, by the continuity ζ and Lemma 2.5, inequality (22) implies that

$$\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0)) \leq \tau_0(d) \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D.$$

It leads to

$$\sup_{\substack{x\in\Omega\\ \theta\in\Theta}} (\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0))) \leq \inf_{d\in D} \tau_0(d),$$

and hence $x_0 \in E(\tau_0, \gamma_0)$. Therefore, by Lemma 2.2(b), E is usc at (τ_0, γ_0). □

Next, in what follows, we consider a case of $D = d_0 + K$, where $d_0 \in \text{int}\, K$, we establish sufficient conditions of the continuity of the map E as follows.

Theorem 4.2 *Assume that assumptions* $(\mathcal{A}'3)$, $(\mathcal{A}4)$ *and* $(\mathcal{A}5)$ *are satisfied. Then, E is continuous and compact valued on* $K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma$.

Proof Due to Theorem 4.1, we only need to prove that E is lsc at every element (τ_0, γ_0) in $K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma$.

◇ We first consider the map $\widehat{E} : K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma \rightrightarrows \Omega$ defined by

$$\begin{aligned}(\tau, \gamma) \mapsto\ & \widehat{E}(\tau, \gamma) \\ := & \left\{ x_0 \in \Omega : \sup_{\substack{x\in\Omega\\ \theta\in\Theta}} (\tau(\zeta(x_0, \theta_0, \gamma)) - \tau(\zeta(x, \theta, \gamma))) < \inf_{d\in D} \tau(d) \right\},\end{aligned}$$

for all $(\tau, \gamma) \in K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma$. We will show that $\widehat{E}$ is lower semicontinuous at every element (τ_0, γ_0) in $K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma$.

If $\widehat{E}$ is not lower semicontinuous at some $(\tau_0, \gamma_0) \in K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma$, then we can find $x_0 \in \widehat{E}(\tau_0, \gamma_0)$ and $\{(\tau_n, \gamma_n)\}$ converging to (τ_0, γ_0) such that for any sequence $\{x_n\}$ with $x_n \in \widehat{E}(\tau_n, \gamma_n)$, $\{x_n\}$ cannot converge to x_0. Consequently, there exists a subsequence $\{(\tau_{n_k}, \gamma_{n_k})\}$ of $\{(\tau_n, \gamma_n)\}$ such that $x_0 \notin \widehat{E}(\tau_{n_k}, \gamma_{n_k})$ for all n_k, or equivalently

$$\sup_{\substack{x\in\Omega\\ \theta\in\Theta}} (\tau_{n_k}(\zeta(x_0, \theta_0, \gamma_{n_k})) - \tau_{n_k}(\zeta(x, \theta, \gamma_{n_k}))) \geq \inf_{d\in D} \tau_{n_k}(d).$$

Then, there exist $(\hat{x}_{n_k}, \hat{\theta}_{n_k}) \in \Omega \times \Theta$ and $\hat{d}_{n_k} \in D$ such that

$$\tau_{n_k}(\zeta(x_0, \theta_0, \gamma_{n_k})) - \tau_{n_k}(\zeta(\hat{x}_{n_k}, \hat{\theta}_{n_k}, \gamma_{n_k})) \geq \tau_{n_k}(\hat{d}_{n_k}). \tag{23}$$

Since $\Omega \times \Theta$ is compact, we can assume that $\{(\hat{x}_{n_k}, \hat{\theta}_{n_k})\}$ converges to $(\hat{x}, \hat{\theta}) \in \Omega \times \Theta$. On the other hand, due to $\hat{d}_{n_k} \in D$, we have $\hat{d}_{n_k} = d_0 + k_{n_k}$ as $k_{n_k} \in K$. This together with (23) helps us to conclude that

$$\tau_{n_k}(\zeta(x_0, \theta_0, \gamma_{n_k})) - \tau_{n_k}(\zeta(\hat{x}_{n_k}, \hat{\theta}_{n_k}, \gamma_{n_k})) \geq \tau_{n_k}(d_0) + \tau_{n_k}(k_{n_k}).$$

By $k_{n_k} \in K$ and $\tau_{n_k} \in K^* \backslash \{0_{\mathbb{Y}^*}\}$, we have $\tau_{n_k}(k_{n_k}) \geq 0$, and hence

$$\tau_{n_k}(\zeta(x_0, \theta_0, \gamma_{n_k})) - \tau_{n_k}(\zeta(\hat{x}_{n_k}, \hat{\theta}_{n_k}, \gamma_{n_k})) \geq \tau_{n_k}(d_0). \tag{24}$$

By the continuity of ζ and τ_{n_k}, and Lemma 2.5, inequality (24) provides that

$$\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(\hat{x}, \hat{\theta}, \gamma_0)) \geq \tau_0(d_0). \tag{25}$$

Moreover, because $x_0 \in \widehat{E}(\tau_0, \gamma_0)$, we have

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} (\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0))) < \inf_{d \in D} \tau_0(d),$$

which implies that

$$\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0)) < \inf_{d \in D} \tau_0(d) \text{ for all } (x, \theta) \in \Omega \times \Theta.$$

This is impossible due to (25). Therefore, $\widehat{E}$ is lower semicontinuous at (τ_0, γ_0).
$\diamond$ Next, for any $\tilde{x} \in E(\tau_0, \gamma_0)$ and $\hat{x} \in \widehat{E}(\tau_0, \gamma_0)$, by the natural K-quasiconvexity of $\zeta(\cdot, \theta_0, \gamma_0)$ on Ω, we conclude that for each $\lambda \in]0, 1[$, there is $\mu \in]0, 1[$ satisfying

$$\zeta\ (x_\lambda, \theta_0, \gamma_0) \in \mu\zeta(\tilde{x}, \theta_0, \gamma_0) + (1 - \mu)\zeta(\hat{x}, \theta_0, \gamma_0) - K,$$

where $x_\lambda = (1 - \lambda)\tilde{x} + \lambda\hat{x}$. Then, we can find some $k \in K$ such that

$$\zeta\ (x_\lambda, \theta_0, \gamma_0) = \mu\zeta(\tilde{x}, \theta_0, \gamma_0) + (1 - \mu)\zeta(\hat{x}, \theta_0, \gamma_0) - k.$$

Combining this with $\tau_0(k) \geq 0$, we obtain

$$\tau_0(\zeta(x_\lambda, \theta_0, \gamma_0)) \leq \mu\tau_0(\zeta(\tilde{x}, \theta_0, \gamma_0)) + (1 - \mu)\tau_0(\zeta(\hat{x}, \theta_0, \gamma_0)).$$

This together with $\tilde{x} \in E(\tau_0, \gamma_0)$ and $\hat{x} \in \widehat{E}(\tau_0, \gamma_0)$ implies that

$$\begin{aligned} \tau_0(\zeta(x_\lambda, \theta_0, \gamma_0)) &< \mu\left[\tau_0(\zeta(x, \theta, \gamma_0)) + \tau_0(d)\right] + (1 - \mu)\left[\tau_0(\zeta(x, \theta, \gamma_0)) + \tau_0(d)\right] \\ &< \tau_0(\zeta(x, \theta, \gamma_0)) + \tau_0(d), \end{aligned}$$

for all $(x, \theta) \in \Omega \times \Theta$ and $d \in D$. Therefore,

$$\tau_0(\zeta(x_\lambda, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0)) < \tau_0(d) \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D. \quad (26)$$

On the other hand, due to the compactness of Ω and Θ, the continuity of ζ and τ_0 will imply that there exists $(\bar{x}, \bar{\theta}) \in \Omega \times \Theta$ such that

$$\tau_0(\zeta(x_\lambda, \theta_0, \gamma_0)) - \tau_0(\zeta(\bar{x}, \bar{\theta}, \gamma_0)) = \sup_{\substack{x \in \Omega \\ \theta \in \Theta}} (\tau_0(\zeta(x_\lambda, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0))). \quad (27)$$

Moreover, because $\tau_0 \in K^* \setminus \{0_{\mathbb{Y}^*}\}$, we have

$$\tau_0(d_0) = \inf_{d \in D} \tau_0(d). \quad (28)$$

It follows from (26), (27) and (28) that

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} (\tau_0(\zeta(\lambda\tilde{x} + (1-\lambda)\hat{x}, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0))) < \inf_{d \in D} \tau_0(d),$$

and hence $x_\lambda \in \widehat{E}(\tau_0, \gamma_0)$ for all $\lambda \in]0, 1[$. Combining this with $x_\lambda = (1-\lambda)\tilde{x} + \lambda\hat{x} \to \tilde{x}$ when $\lambda \to 0$, we get that $\tilde{x} \in \operatorname{cl} \widehat{E}(\tau_0, \gamma_0)$, and hence $E(\tau_0, \gamma_0) \subseteq \operatorname{cl} \widehat{E}(\tau_0, \gamma_0)$. Since $\widehat{E}$ is lower semicontinuous at (τ_0, γ_0), we obtain

$$E(\tau_0, \gamma_0) \subseteq \operatorname{cl} \widehat{E}(\tau_0, \gamma_0) \subseteq \liminf \widehat{E}(\tau_n, \gamma_n) \subseteq \liminf E(\tau_n, \gamma_n),$$

for any sequence $\{(\tau_n, \gamma_n)\}$ converging to (τ_0, γ_0). Consequently, E is lower semicontinuous at (τ_0, γ_0), and hence the proof is finished. □

The following result dedicates sufficient conditions for the semicontinuities of robust Benson efficient solution map of (PUVOP) corresponding to the scenario θ_0.

Theorem 4.3 *Assume that assumptions* $(\mathcal{A}0)$, $(\mathcal{A}'1)$, $(\mathcal{A}'3)$, $(\mathcal{A}4)$ *and* $(\mathcal{A}5)$ *are fulfilled. Then,* $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)$ *is lower semicontinuous and Hausdorff upper semicontinuous with nonempty values on* Γ.

Proof By Lemma 4.1, we obtain

$$\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma) = \bigcup_{\tau \in K^\#} E(\tau, \gamma) \text{ for all } \gamma \in \Gamma. \quad (29)$$

Due to the continuity of ζ and τ, the map $\tau \circ \zeta$ is continuous. Combining this with the compactness of Ω, we deduce that $E(\tau, \gamma)$ is nonempty, and hence $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma)$ is also nonempty. Moreover, for any $\tau \in K^\#$, Theorem 4.2 implies that $E(\tau, \cdot)$ is lower semicontinuous on Γ. This together with Lemma 2.4 yields that $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)$ is lower semicontinuous on Γ.

To prove that $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)$ is Hausdorff upper semicontinuous on Γ, we consider the map $W : \Gamma \rightrightarrows \Omega$ defined by

$$W(\gamma) := \bigcup_{\tau \in K^* \setminus \{0_{\mathbb{Y}^*}\}} E(\tau, \gamma) \text{ for all } \gamma \in \Gamma.$$

Building on this map, the desired conclusion will be derived via the following steps.
Step 1: We claim that

$$\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma) \subseteq W(\gamma) \subseteq \mathrm{cl}\,\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma) \text{ for all } \gamma \in \Gamma. \tag{30}$$

By (29) and the inclusion $\bigcup_{\tau \in K^\#} E(\tau, \gamma) \subseteq \bigcup_{\tau \in K^* \setminus \{0_{\mathbb{Y}^*}\}} E(\tau, \gamma)$ for all $\gamma \in \Gamma$, the first inclusion of (30) is true.

Moreover, for any $\hat{\tau} \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ and $\bar{\tau} \in K^\#$, we have

$$\hat{\tau}(k) \geq 0 \text{ and } \bar{\tau}(k) > 0 \text{ for all } k \in K.$$

For $n \in \mathbb{N}$, we set

$$\tau_n(k) = \hat{\tau}(k) + \frac{1}{n}\bar{\tau}(k) \text{ for all } k \in K.$$

Then, $\tau_n(k) > 0$ for all $k \in K \setminus \{0_{\mathbb{Y}}\}$, and so $\tau_n \in K^\#$. Furthermore, $\tau_n \to \hat{\tau}$ when $n \to +\infty$. It means that $\hat{\tau} \in \mathrm{cl}\, K^\#$, and hence

$$K^* \setminus \{0_{\mathbb{Y}^*}\} \subset \mathrm{cl}\, K^\#. \tag{31}$$

Now, we are in a position to present the proof of the second inclusion in (30). For each $\gamma \in \Gamma$, taking an arbitrary element $x_0 \in W(\gamma)$, we can find $\hat{\tau}_0 \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ such that $x_0 \in E(\hat{\tau}_0, \gamma)$. Due to (31), there exists $\{\hat{\tau}_n\} \subset K^\#$ converging to $\hat{\tau}_0$. By applying Theorem 4.2, E is lower semicontinuous on $K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma$, and so $E(\cdot, \gamma)$ is also lower semicontinuous on $K^* \setminus \{0_{\mathbb{Y}^*}\}$. Consequently, there exist $x_n \in E(\hat{\tau}_n, \gamma) \subseteq \bigcup_{\tau \in K^\#} E(\tau, \gamma)$ such that the sequence $\{x_n\}$ converges to x_0, which implies that

$$x_0 \in \mathrm{cl} \bigcup_{\tau \in K^\#} E(\tau, \gamma) = \mathrm{cl}\,\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma).$$

Therefore, $W(\gamma) \subseteq \mathrm{cl}\,\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma)$ for all $\gamma \in \Gamma$.
Step 2: W is upper semicontinuous on Γ.

Let $\gamma_0 \in \Gamma$ be arbitrary. Suppose that this is not the case, there exist an open set Δ with $W(\gamma_0) \subseteq \Delta$ and a sequence $\{\gamma_n\} \subset \Gamma$ with $\gamma_n \to \gamma_0$ such that there exists $x_n \in W(\gamma_n) \setminus \Delta$ for any $n \in \mathbb{N}$. By the definition of W, we can get $\tau_n \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ such that $x_n \in E(\tau_n, \gamma_n)$. In view of the definition of the map E, we have

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} (\tau_n(\zeta(x_n, \theta_0, \gamma_n)) - \tau_n(\zeta(x, \theta, \gamma_n))) \leq \inf_{d \in D} \tau_n(d),$$

which implies that

$$\tau_n(\zeta(x_n, \theta_0, \gamma_n)) - \tau_n(\zeta(x, \theta, \gamma_n)) \le \tau_n(d) \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D.$$

It leads to

$$\tau_n\left(d - \zeta(x_n, \theta_0, \gamma_n) + \zeta(x, \theta, \gamma_n)\right) \ge 0 \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D.$$

Combining this with $\tau_n \in K^* \setminus \{0_{\mathbb{Y}^*}\}$, we get

$$d - \zeta(x_n, \theta_0, \gamma_n) + \zeta(x, \theta, \gamma_n) \notin -\operatorname{int} K \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D. \quad (32)$$

Since $\{x_n\}$ is a sequence within the compact set Ω, it can be assumed that $\{x_n\}$ converges to some x_0. By the continuity of ζ on $\Omega \times \Theta \times \Gamma$ and the closeness of $\mathbb{Y} \setminus (-\operatorname{int} K)$, statement (32) yields that

$$d - \zeta(x_0, \theta_0, \gamma_0) + \zeta(x, \theta, \gamma_0) \notin -\operatorname{int} K \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D.$$

This together with equality (1) and $(\mathcal{A}'1)$ implies that there exists $\tau_0 \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ such that

$$\tau_0\left(d - \zeta(x_0, \theta_0, \gamma_0) + \zeta(x, \theta, \gamma_0)\right) \ge 0 \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D,$$

or equivalently

$$\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0)) \le \tau_0(d) \text{ for all } (x, \theta, d) \in \Omega \times \Theta \times D.$$

Therefore,

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left(\tau_0(\zeta(x_0, \theta_0, \gamma_0)) - \tau_0(\zeta(x, \theta, \gamma_0))\right) \le \inf_{d \in D} \tau_0(d),$$

and hence $x_0 \in E(\tau_0, \gamma_0) \subseteq W(\gamma_0) \subseteq \Delta$. This is impossible as $x_n \notin \Delta$ for all n. Thus, W is upper semicontinuous on Γ.

Step 3: $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)$ is Hausdorff upper semicontinuous on Γ.

Let $\gamma_0 \in \Gamma$ be arbitrary. For any neighborhood Ξ of the origin in $\mathbb{Y}$, there exists a balance neighborhood Ξ_1 of the origin in $\mathbb{Y}$ such that

$$\Xi_1 + \Xi_1 \subseteq \Xi.$$

Since W is upper semicontinuous at γ_0, W is also Hausdorff upper semicontinuous at γ_0. Hence, we can pick up a neighborhood Δ of γ_0 such that

$$W(\gamma) \subseteq W(\gamma_0) + \Xi_1 \text{ for all } \gamma \in \Delta. \quad (33)$$

Since $W(\gamma_0) \subseteq \mathrm{cl}\,\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma_0)$ and $\Xi_1 + \Xi_1 \subseteq \Xi$, the inclusion (33) implies that

$$\begin{aligned} W(\gamma) \subseteq W(\gamma_0) + \Xi_1 &\subseteq \mathrm{cl}\,\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma_0) + \Xi_1 \\ &\subseteq \mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma_0) + \Xi_1 + \Xi_1 \\ &\subseteq \mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma_0) + \Xi, \end{aligned}$$

for all $\gamma \in \Delta$. This together with (30) helps us to get that

$$\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma) \subseteq W(\gamma) \subseteq \mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma_0) + \Xi \quad \text{for all } \gamma \in \Delta.$$

Therefore, $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)$ is Husc at every element γ_0 in Γ. □

When $\mathbb{Y}$ is a finite dimensional space, by applying Theorem 4.3, we obtain sufficient conditions of the semicontinuities for the map $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)$ as follows.

Corollary 4.1 *Assume that assumptions $(\mathcal{A}'1)$, $(\mathcal{A}'3)$, $(\mathcal{A}4)$ and $(\mathcal{A}5)$ hold. Then,* $\mathrm{BEff}_K(\mathrm{PUVOP})$ (ζ, D) *is lower semicontinuous and Hausdorff upper semicontinuous with nonempty values on Γ.*

Example 4.1 Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^2, \mathbb{P} = \mathbb{R}, \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 2, 0 \le x_2 \le 2\}$, $K = \mathbb{R}^2_+$, $D = (1, 1) + \mathbb{R}^2_+$, $\Gamma = [-1, 1]$, $\Theta = [-1, 1]$, $\theta_0 = 1$ and $\zeta : \Omega \times [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ be defined by

$$\zeta(\boldsymbol{x}, \theta, \gamma) := e^{\gamma} \left(x_1 + 2|\theta|, x_1^2 + 2x_2^2 + 3\theta^2 - 1 \right) \text{ for all } \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2.$$

It is easy to check that all the assumptions of Corollary 4.1 are satisfied. By applying Corollary 4.1, we conclude that $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)$ is lower semicontinuous and Hausdorff upper semicontinuous on Γ.

Passing to the semicontinuity of robust Benson efficient solution map of the parametric constrained uncertain vector optimization problem, for each $\gamma \in \Gamma$, we focus on the following problem.

$$(\mathrm{PCUVOP}) \qquad \min_{\substack{x \in \Omega, \theta \in \Theta \\ \eta(x,\theta) \in -C}} \zeta(x, \theta, \gamma),$$

where $\zeta : \Omega \times \Theta \times \Gamma \to \mathbb{Y}$ and $\eta : \Omega \times \Theta \to \mathbb{W}$ be vector-valued maps. Similar to the previous section, we also define

$$\Sigma := \{(x, \theta) \in \Omega \times \Theta : \eta(x, \theta) \in -C\},$$

and propose the concept of robust Benson efficient solution of (PCUVOP) as follows.

Definition 4.2 Let $\theta_0 \in \Theta$ be given. For each $\gamma \in \Gamma$, an element $x_0 \in \Omega$ is called *a robust Benson efficient solution* of (PCUVOP) corresponding to the scenario $\theta_0 \in \Theta$, written as $x_0 \in \mathrm{BEff}_K(\mathrm{PCUVOP})(\zeta, \eta, D)(\gamma)$, if

$$\operatorname{clcone}\left(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : (x, \theta) \in \Sigma\} + D\right) \cap (-K) = \{0_{\mathbb{Y}}\}.$$

We are now discussing a critical result that plays an important role in studying the stability of the problem (PCUVOP).

Lemma 4.2 *Assume that assumptions* $(\mathcal{A}'3)$ *and* $(\mathcal{A}6)$ *are satisfied. Then, the set* Σ *is convex and compact.*

Proof Firstly, let $(x_1, \theta_1), (x_2, \theta_2) \in \Sigma$ and $\lambda \in [0, 1]$ be arbitrary. Because of the natural C-quasiconvexity of η, there exists $\mu \in]0, 1[$ satisfying

$$\eta\left(\lambda(x_1, \theta_1) + (1 - \lambda)(x_2, \theta_2)\right) \in \mu\eta(x_1, \theta_1) + (1 - \mu)\eta(x_2, \theta_2) - C. \tag{34}$$

On the other hand, due to $(x_1, \theta_1), (x_2, \theta_2) \in \Sigma$, we have

$$\eta(x_1, \theta_1) \in -C \text{ and } \eta(x_2, \theta_2) \in -C,$$

and consequently

$$\mu\eta(x_1, \theta_1) \in -C \text{ and } (1 - \mu)\eta(x_2, \theta_2) \in -C,$$

as C is a cone and $\mu \in [0, 1]$. These together with (34) and the convexity of $(-C)$ imply that

$$\eta\left(\lambda(x_1, \theta_1) + (1 - \lambda)(x_2, \theta_2)\right) \in -C - C - C \subseteq -C.$$

It leads to $\lambda(x_1, \theta_1) + (1 - \lambda)(x_2, \theta_2) \in \Sigma$, and hence Σ is convex.

Next, due to assumption $(\mathcal{A}'3)$, we will obtain the compactness of Σ through its closed property. Suppose that there exists $\{(x_n, \theta_n)\} \subset \Sigma$ such that $(x_n, \theta_n) \to (\hat{x}, \hat{\theta})$ but $\eta(\hat{x}, \hat{\theta}) \notin -C$, namely $\eta(\hat{x}, \hat{\theta}) \in \mathbb{Y} \setminus (-C)$. Since C is closed, $\mathbb{Y} \setminus (-C)$ is an open neighborhood of $\eta(\hat{x}, \hat{\theta})$. Combining this with the C-lower semicontinuity of η at $(\hat{x}, \hat{\theta})$, we conclude that

$$\eta(x_n, \theta_n) \in (\mathbb{Y} \setminus (-C)) + C,$$

and hence there exist $y_n \in \mathbb{Y} \setminus (-C)$, $c_n \in C$ such that

$$\eta(x_n, \theta_n) = y_n + c_n. \tag{35}$$

Moreover, it follows from $(x_n, \theta_n) \in \Sigma$ that $\eta(x_n, \theta_n) \in -C$, and so for each $n \in \mathbb{N}$, there exists $\hat{c}_n \in C$ such that $\eta(x_n, \theta_n) = -\hat{c}_n$. This together with (35) implies that

$$y_n = \eta(x_n, \theta_n) - c_n = -\hat{c}_n - c_n \in -C - C \subseteq -C,$$

which is impossible as $y_n \in \mathbb{Y} \setminus (-C)$. Consequently, $\eta(\hat{x}, \hat{\theta}) \in -C$, and thus Σ is compact. □

Thanks to Theorems 3.3, 3.4, 4.3 and Lemma 4.2, we establish stability conditions for robust Benson efficient solutions of the parametric constrained uncertain vector optimization problem given in the following result.

Theorem 4.4 *Assume that assumptions* $(\mathcal{A}0)$, $(\mathcal{A}'1)$, $(\mathcal{A}2)$, $(\mathcal{A}'3)$, $(\mathcal{A}4)$, $(\mathcal{A}5)$ *and* $(\mathcal{A}6)$ *are satisfied. Then,* $\mathrm{BEff}_K(\mathrm{PCUVOP})(\zeta, \eta, D)$ *is lower semicontinuous and Hausdorff upper semicontinuous with nonempty values on* Γ.

In the case where $\mathbb{Y}$ is a finite-dimensional space, the stability of the robust Benson efficient solution for the parametric constrained uncertain vector optimization problem can be deduced from the above theorem, as stated in the following result.

Corollary 4.2 *Assume that assumptions* $(\mathcal{A}'1)$, $(\mathcal{A}2)$, $(\mathcal{A}'3)$, $(\mathcal{A}4)$-$(\mathcal{A}6)$ *are fulfilled. Then,* BEff_K $(\mathrm{PCUVOP})(\zeta, \eta, D)$ *is lower semicontinuous and Hausdorff upper semicontinuous with nonempty values on* Γ.

Example 4.2 Let $\mathbb{M}^n$ be a space of all the real $n \times n$ symmetric matrices, $\mathbb{X} \subset \mathbb{M}^n$ is a set of symmetric matrices with negative eigenvalues, and $\Omega \subset \mathbb{X}$ is closed and bounded, that is there exists $q \in \mathbb{R}_+$ satisfying

$$\|X\| := \sup_{\|z\|=1} \|Xz\| \leq q \;\text{ for all } X \in \Omega,$$

where $\|w\| := \operatorname*{Max}_{i\in\{1,\cdots,n\}} |w_i|$ for $w = (w_1, \cdots, w_n) \in \mathbb{R}^n$.

Then, Ω is compact as $\mathbb{M}^n$ is a finite dimensional space. We will show that Ω is convex. Let $X_1, X_2 \in \Omega$ and $t \in [0, 1]$ be arbitrary, we have then

$$\|tX_1 + (1-t)X_2\| \leq t\|X_1\| + (1-t)\|X_2\| \leq q.$$

On the other hand, tX_1 and $(1-t)X_2$ have the same eigenvalues as X_1 and X_2, respectively. Therefore, $tX_1 + (1-t)X_2$ will also have negative eigenvalues and consequently $tX_1 + (1-t)X_2 \in \Omega$.

Let $\mathbb{P} = \mathbb{R}$, $\mathbb{Y} = \mathbb{W} = \mathbb{R}^2$, $K = C = \mathbb{R}^2_+$, $D = (1, 1) + \mathbb{R}^2_+$, $\Theta = [-1, 1]$, $\theta_0 = 0$ and $\Gamma = [0, 1]$. We define $\zeta : \mathbb{X} \times \Theta \times \Gamma \to \mathbb{R}^2$ by

$$\zeta(X, \theta, \gamma) := \begin{cases} (2^{\gamma}|\det X|, |\theta|) & \text{if } X \in \Omega, \\ (0, 0) & \text{otherwise.} \end{cases}$$

Now let $\eta : \Omega \times \Theta \to \mathbb{R}^2$ be defined by

$$\eta(X, \theta) := \begin{cases} -((-\theta)^{1/2}, |\det X|) & \text{if } \theta \leqslant 0, \\ (\theta^{1/2}, -|\det X|) & \text{if } \theta > 0. \end{cases}$$

Based on the same techniques in the proof of Theorem 1 in [54], we also obtain that ζ is continuous on $\Omega \times \Theta \times \Gamma$, and ζ is naturally K-quasiconvex in the first variable on Ω. Moreover, for each $\gamma \in \Gamma$, by direct computations, we obtain the near D-subconvexlikeness of $\zeta(\cdot, \cdot, \gamma)$ on $\Omega \times \Theta$.

Finally, Θ is compact and η is C-lower semicontinuous as well as naturally C-quasiconvex on $\Omega \times \Theta$. Furthermore, for any $(X, \theta) \in \Omega \times [-1, 0[$, we have $\eta(X, \theta) \in -\operatorname{int}\mathbb{R}^2_+$, and hence all the assumptions of Corollary 4.2 are satisfied. Therefore, by applying Corollary 4.2, we conclude that $\operatorname{BEff}_K(\text{PCUVOP})(\zeta, \eta, D)$ is lower semicontinuous and Hausdorff upper semicontinuous on Γ.

In fact, it is easy to see that $\Sigma = \{(X, \theta) : X \in \Omega, -1 \leq \theta \leq 0\} = \Omega \times [-1, 0]$. For each $\gamma \in \Gamma$, $X, X_0 \in \Omega$, and $\theta \in \Theta$, we have

$$\zeta(X, \theta, \gamma) - \zeta(X_0, \theta_0, \gamma) + D = \big(2^\gamma(|\det X| - |\det X_0|) + 1, |\theta| + 1\big) + \mathbb{R}^2_+.$$

Setting

$$\mathcal{A} := \left\{\big(2^\gamma(|\det X| - |\det X_0|) + 1, |\theta| + 1\big) : (X, \theta) \in \Sigma\right\}.$$

Because $0 \leq |\theta| \leq 1$, we have $1 \leq |\theta| + 1 \leq 2$. Moreover, since $\|X\| \leq q$, all the entries of X are bounded. Due to the Leibniz formula for determinants and $\gamma \in [0, 1]$, we conclude that there exists $m, M \in \mathbb{R}$ satisfying

$$m \leq 2^\gamma(|\det X| - |\det X_0|) + 1 \leq M \text{ for all } X \in \Omega.$$

Thus, through direct computation, we get that

$$\operatorname{clcone}(\mathcal{A} + \mathbb{R}^2_+) \cap (-K) = \{0_{\mathbb{Y}}\} \text{ for all } X_0 \in \Omega,$$

and hence $\operatorname{BEff}_K(\text{PCUVOP})(\zeta, \eta, D)(\gamma) = \Omega$ for all $\gamma \in \Gamma$.

5 Hausdorff Well-posedness for Parametric Uncertain Vector Optimization Problems

In this section, we aim to study the Hausdorff well-posedness for (PUVOP). Let $\mathbb{X}, \mathbb{Y}, \mathbb{W}, \mathbb{U}, \mathbb{P}, K, C, \Omega, \Theta, D, \Gamma, \zeta, \eta, \Sigma$ be defined as in Sect. 4, and let $k_0 \in K \setminus \{0_{\mathbb{Y}}\}$ be given. Motivated by [20, 23, 58, 59], we consider approximate robust Benson efficient solutions of (PUVOP) as follows.

Definition 5.1 For each $(\gamma, \varepsilon) \in \Gamma \times \mathbb{R}_+$, an element $x_0 \in \Omega$ is called an *ε-robust Benson efficient solution* of (PUVOP) corresponding to $\theta_0 \in \Theta$, written as $x_0 \in \widehat{\operatorname{BEff}}_K(\text{PUVOP})(\zeta, D)(\gamma, \varepsilon)$, if

$$\operatorname{clcone}\left(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : (x, \theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right) \cap (-K) = \{0_{\mathbb{Y}}\}.$$

Picking up the ideas of [20, 23, 25, 58], we also discuss the following properties of the approximate solution sets.

Lemma 5.1 *The following statements are true.*

(a) *For every $\gamma \in \varGamma$,*

$$\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma) = \widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, 0).$$

(b) *For every $\gamma \in \varGamma$ and $0 \leq \varepsilon_1 \leq \varepsilon_2$,*

$$\widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, \varepsilon_1) \subseteq \widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, \varepsilon_2).$$

Proof Statement (a) is implied directly from the definition of the approximate solution set.

In order to prove statement (b), we first provide the following inclusion:

$$\varepsilon_2 k_0 + D = \varepsilon_1 k_0 + (\varepsilon_2 - \varepsilon_1) k_0 + D \subseteq \varepsilon_1 k_0 + D.$$

This inclusion is true because of $(\varepsilon_2 - \varepsilon_1) k_0 + D \subseteq K + D = D$. Therefore, the set

$$\mathrm{clcone}\left(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : (x, \theta) \in \varOmega \times \varTheta\} + \varepsilon_2 k_0 + D\right)$$

is a subset of $\mathrm{clcone}\left(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : (x, \theta) \in \varOmega \times \varTheta\} + \varepsilon_1 k_0 + D\right)$. Consequently, if

$$\mathrm{clcone}\left(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : (x, \theta) \in \varOmega \times \varTheta\} + \varepsilon_1 k_0 + D\right) \cap (-K) = \{0_{\mathbb{Y}}\},$$

then

$$\mathrm{clcone}\left(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : (x, \theta) \in \varOmega \times \varTheta\} + \varepsilon_2 k_0 + D\right) \cap (-K) = \{0_{\mathbb{Y}}\},$$

or equivalently $\widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, \varepsilon_1) \subseteq \widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, \varepsilon_2)$. □

Next, inspired by Definition 9.1.2 in [3] and [49], we define the Hausdorff well-posedness for the problem (PUVOP) for robust Benson efficient solutions.

Definition 5.2 The problem (PUVOP) is said to be Hausdorff well-posed on $\varGamma$ for robust Benson efficient solutions (in short, Hausdorff well-posed on $\varGamma$) if for any $\gamma \in \varGamma$,

(a) $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma) \neq \emptyset$,
(b) the solution map $\widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)$ is upper Hausdorff semicontinuous at $(\gamma, 0)$.

Similar to Sect. 3, we consider the map $S : K^* \times \varGamma \times \mathbb{R}_+ \rightrightarrows X$ defined by

$$\begin{aligned}(\tau, \gamma, \varepsilon) &\mapsto S(\tau, \gamma, \varepsilon)\\ &:= \left\{ x_0 \in \varOmega : \sup_{\substack{x \in \varOmega \\ \theta \in \varTheta}} \left(\tau(\zeta(x_0, \theta_0, \gamma)) - \tau(\zeta(x, \theta, \gamma))\right) \leq \varepsilon \tau(k_0) + \inf_{d \in D} \tau(d) \right\},\end{aligned}$$

for all $(\tau, \gamma, \varepsilon) \in K^* \times \varGamma \times \mathbb{R}_+$.

Lemma 5.2 *For each* $(\gamma, \varepsilon) \in \Gamma \times \mathbb{R}_+$, *if assumptions* $(\mathcal{A}0)$ *and* $(\mathcal{A}'1)$ *are satisfied, then*

$$\widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, \varepsilon) = \bigcup_{\tau \in K^\#} S(\tau, \gamma, \varepsilon). \tag{36}$$

Proof $(\subseteq)$ Let $x_0 \in \widehat{\mathrm{BEff}}_K(\mathrm{UVOP})(\zeta, D)(\gamma, \varepsilon)$ be arbitrary. It follows from the definition of the ε-robust Benson efficient solution that

$$\operatorname{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right) \cap (-K) = \{0_{\mathbb{Y}}\}. \tag{37}$$

In view of assumption $(\mathcal{A}'1)$, we imply that

$$\operatorname{clcone}\left(\{\zeta(x,\theta,\gamma) : (x,\theta) \in \Omega \times \Theta\} + D\right)$$

is a convex set, and so the set $\operatorname{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right)$ is convex. Together with (37) and assumption $(\mathcal{A}0)$, Lemma 2.6(a) implies that there exists $\hat{\tau} \in K^\#$ satisfying

$$\hat{\tau} \in \left(\operatorname{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right)\right)^*,$$

which leads to

$$\hat{\tau}(z) \geq 0 \text{ for all } z \in \operatorname{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right).$$

Consequently, $\hat{\tau}(z) \geq 0$ for all $z \in \{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D$. Hence,

$$\hat{\tau}(\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) + \varepsilon k_0 + d) \geq 0 \text{ for all } (x,\theta,d) \in \Omega \times \Theta \times D,$$

this provides that

$$\hat{\tau}(\zeta(x_0,\theta_0,\gamma)) - \hat{\tau}(\zeta(x,\theta,\gamma)) \leq \varepsilon\hat{\tau}(k_0) + \tau(d) \text{ for all } (x,\theta,d) \in \Omega \times \Theta \times D.$$

Therefore,

$$\sup_{\substack{x \in \Omega \\ \theta \in \Theta}} \left(\hat{\tau}(\zeta(x_0,\theta_0,\gamma)) - \hat{\tau}(\zeta(x,\theta,\gamma))\right) \leq \varepsilon\hat{\tau}(k_0) + \inf_{d \in D} \hat{\tau}(d) \text{ for all } (x,\theta) \in \Omega \times \Theta.$$

Thus, $x_0 \in S(\hat{\tau}, \gamma, \varepsilon) \subseteq \bigcup_{\tau \in K^\#} S(\tau, \gamma, \varepsilon)$.

$(\supseteq)$ Taking an arbitrary element $x_0 \in \bigcup_{\tau \in K^\#} S(\tau, \gamma, \varepsilon)$, there exists $\bar{\tau} \in K^\#$ such that $x_0 \in S(\bar{\tau}, \gamma, \varepsilon)$, and so we have

$$\bar{\tau}(\zeta(x_0,\theta_0,\gamma)) - \bar{\tau}(\zeta(x,\theta,\gamma)) \leq \varepsilon\bar{\tau}(k_0) + \bar{\tau}(d) \text{ for all } (x,\theta,d) \in \Omega \times \Theta \times D. \tag{38}$$

Moreover, we also have

$$0_{\mathbb{Y}} \in \text{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right) \cap (-K).$$

Now we will show that for any element

$$a \in \text{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right) \cap (-K),$$

it follows that $a = 0_{\mathbb{Y}}$. Since D is a free disposal set w.r.t. K, we obtain

$$a \in \text{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D + K\right) \cap (-K).$$

It follows from $a \in \text{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D + K\right)$ that there exist sequences $\{t_n\} \subset \mathbb{R}_+$, $\{(x_n,\theta_n)\} \subset \Omega \times \Theta$, $\{d_n\} \subset D$ and $\{v_n\} \subset K$ such that

$$t_n(\zeta(x_n,\theta_n,\gamma) - \zeta(x_0,\theta_0,\gamma) + \varepsilon k_0 + d_n + v_n) \to a.$$

Because $\hat{\tau} \in K^{\#}$, we derive that

$$t_n\left(\bar{\tau}(\zeta(x_n,\theta_n,\gamma)) - \bar{\tau}(\zeta(x_0,\theta_0,\gamma)) + \varepsilon\bar{\tau}(k_0) + \bar{\tau}(d_n) + \bar{\tau}(v_n)\right) \to \bar{\tau}(a). \tag{39}$$

Moreover, since $\tau \in K^{\#}$ and $v_n \in K$, we get

$$\bar{\tau}(v_n) \geq 0. \tag{40}$$

By (38), we obtain $\bar{\tau}(\zeta(x_0,\theta_0,\gamma)) - \bar{\tau}(\zeta(x_n,\theta_n,\gamma)) \leq \varepsilon\bar{\tau}(k_0) + \bar{\tau}(d_n)$, which implies that

$$\bar{\tau}(\zeta(x_n,\theta_n,\gamma)) - \bar{\tau}(\zeta(x_0,\theta_0,\gamma)) + \varepsilon\bar{\tau}(k_0) + \bar{\tau}(d_n) \geq 0. \tag{41}$$

Due to (39), (40) and (41), we conclude that

$$\bar{\tau}(a) \geq 0. \tag{42}$$

Furthermore, because $\bar{\tau} \in K^{\#}$ and $a \in -K$, we have $\bar{\tau}(a) \leq 0$. Combining this with (42), we obtain that $\bar{\tau}(a) = 0$. Since $\bar{\tau} \in K^{\#}$, we get that $a = 0_{\mathbb{Y}}$, and thus

$$\text{clcone}\left(\{\zeta(x,\theta,\gamma) - \zeta(x_0,\theta_0,\gamma) : (x,\theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\right) \cap (-K) = \{0_{\mathbb{Y}}\}.$$

Therefore, $x_0 \in \widehat{\text{BEff}}_K\,(\text{UVOP})(\zeta, D)(\gamma, \varepsilon)$. □

The result on the scalar representation obtained in Lemma 5.2 plays a very important role in studying the qualitative properties of the solutions for the original problem. Naturally, to achieve this, the auxiliary solution sets $S(\tau,\gamma,\varepsilon)$ must also possess the corresponding solution properties that we need to examine for the original problem. Below are the results regarding the properties of the auxiliary solution sets in the research direction just mentioned.

Lemma 5.3 *Assume that assumptions* $(\mathcal{A}'3)$, $(\mathcal{A}4)$ *and* $(\mathcal{A}5)$ *are true. Then,* $S(\cdot, \gamma, 0)$ *is lower semicontinuous on* $K^* \setminus \{0_{\mathbb{Y}^*}\}$ *for each* $(\gamma, 0) \in \Gamma \times \mathbb{R}_+$.

Proof We first consider the map $\widehat{S} : K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma \times \mathbb{R}_+ \rightrightarrows \Omega$ defined by

$$(\tau, \gamma, \varepsilon) \mapsto \widehat{S}(\tau, \gamma, \varepsilon) := \left\{ x_0 \in \Omega : \sup_{\substack{x \in \Omega \\ \theta \in \Theta}} (\tau(\zeta(x_0, \theta_0, \gamma)) - \tau(\zeta(x, \theta, \gamma))) < \varepsilon\tau(k_0) + \inf_{d \in D} \tau(d) \right\},$$

for all $(\tau, \gamma, \varepsilon) \in K^* \setminus \{0_{\mathbb{Y}^*}\} \times \Gamma \times \mathbb{R}_+$. Then, by using the techniques given in the proof of Theorem 4.2, we also conclude that $\widehat{S}$ is lower semicontinuous in the first variable.

Furthermore, together with assumption $(\mathcal{A}5)$ and the techniques used in the proof of Theorem 4.2, applied again with suitable adjustments, we achieve that for any $\tilde{x} \in S(\tau_0, \gamma, 0)$, $\hat{x} \in \widehat{S}(\tau_0, \gamma, 0)$, and $\lambda \in [0, 1]$, we get that $x_\lambda := (1 - \lambda)\tilde{x} + \lambda\hat{x} \in \widehat{S}(\tau_0, \gamma, 0)$. Combining this with the lower semicontinuity of $\widehat{S}$ at $(\tau_0, \gamma, 0)$, we obtain

$$S(\tau_0, \gamma, 0) \subseteq \operatorname{cl} \widehat{S}(\tau_0, \gamma, 0) \subseteq \liminf \widehat{S}(\tau_n, \gamma, 0) \subseteq \liminf S(\tau_n, \gamma, 0),$$

for any sequence $\{\tau_n\}$ satisfying $\tau_n \to \tau_0$. Therefore, $S(\cdot, \gamma, 0)$ is lower semicontinuous at τ_0. □

Also, building on the ideas from the previous section, we consider the map $\Upsilon : \Gamma \times \mathbb{R}_+ \rightrightarrows \Omega$ defined by

$$\Upsilon(\gamma, \varepsilon) := \bigcup_{\tau \in K^* \setminus \{0_{\mathbb{Y}^*}\}} S(\tau, \gamma, \varepsilon) \text{ for all } (\gamma, \varepsilon) \in \Gamma \times \mathbb{R}_+. \tag{43}$$

We then use this map to study the Hausdorff well-posedness of the problem (PUVOP). In doing so, we first examine the following property of the map Υ.

By employing the techniques from Steps 1 and 2 of the proof of Theorem 4.3, with appropriate adjustments, we obtain the following result.

Lemma 5.4 *The following statements are true.*

(a) *If* $(\mathcal{A}3)$ *and* $(\mathcal{A}4)$ *are fulfilled, then* $\Upsilon(\cdot, 0)$ *is upper semicontinuous on* Γ.
(b) *If assumptions* $(\mathcal{A}'3)$, $(\mathcal{A}4)$ *and* $(\mathcal{A}5)$ *are satisfied, then*

$$\widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, 0) \subseteq \Upsilon(\gamma, 0) \subseteq \operatorname{cl} \widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)(\gamma, 0) \text{ for all } \gamma \in \Gamma.$$

We are now in a position to present the main result of this section, namely sufficient conditions of the Hausdorff well-posedness for (PUVOP).

Theorem 5.1 *Assume that assumptions* $(\mathcal{A}0)$, $(\mathcal{A}'1)$, $(\mathcal{A}'3)$, $(\mathcal{A}4)$ *and* $(\mathcal{A}5)$ *are satisfied. Then, the problem* (PUVOP) *is Hausdorff well-posed on* Γ.

Proof By applying Theorem 4.3, we derive that $\mathrm{BEff}_K(\mathrm{PUVOP})(\zeta, D)(\gamma)$ is also nonempty.

Moreover, by using similar techniques as in the proof of Step 3 of Theorem 4.3, with suitable adjustments, and within the framework of the established results from Lemmas 5.3, and 5.4, we also conclude that $\widehat{\mathrm{BEff}}_K(\mathrm{PUVOP})(\zeta, D)$ is Hausdorff upper semicontinuous at $(\gamma, 0)$. The proof follows. □

To illustrate the applicability of Theorem 5.1, we consider the following example.

Example 5.1 Let X be a given $n \times n$ symmetric and positive semidefinite matrix, $\mathbb{X} = \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ be a closed bounded convex set, $\mathbb{U} = \mathbb{P} = \mathbb{R}$, $\mathbb{Y} = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $D = (0, 1) + \mathbb{R}^2_+$, $\Gamma = [0, 1]$, $\Theta = [0, 2]$, $\theta_0 = 1$ and $\zeta : \Omega \times [0, 2] \times [0, 1] \to \mathbb{R}^2$ be defined by

$$\zeta(x, \theta, \gamma) := \Big(x^T Xx + b^T x, \theta + \gamma\Big),$$

where $b \in \mathbb{R}^n$ is given. Obviously, assumptions $(\mathcal{A}0)$, $(\mathcal{A}'1)$, $(\mathcal{A}'3)$ and $(\mathcal{A}4)$ of Theorem 5.1 are satisfied. Next, by the same techniques of the proof of Corollary 3.3 in [41], ζ is naturally K-quasiconvex in the first component on Ω, and so assumption $(\mathcal{A}5)$ of Theorem 5.1 is also satisfied. Therefore, all the assumptions of Theorem 5.1 are fulfilled, and hence due to this theorem, the problem (PUVOP) is Hausdorff well-posed.

Similar to the previous section, we also study Hausdorff well-posedness for the parametric constrained uncertain vector optimization problem (PCUVOP). Motivated by Definition 5.1, we define the concept of approximate robust Benson efficient solutions of (PCUVOP).

Definition 5.3 For each $(\gamma, \varepsilon) \in \Gamma \times \mathbb{R}_+$, an element $x_0 \in \Omega$ is called an *ε-robust Benson efficient solution* of (PCUVOP) corresponding to $\theta_0 \in \Theta$, written as $x_0 \in \widehat{\mathrm{BEff}}_K(\mathrm{PCUVOP})(\zeta, \eta, D)(\gamma, \varepsilon)$, if

$$\mathrm{clcone}\Big(\{\zeta(x, \theta, \gamma) - \zeta(x_0, \theta_0, \gamma) : \eta(x, \theta) \in -C, (x, \theta) \in \Omega \times \Theta\} + \varepsilon k_0 + D\Big) \cap (-K) = \{0_{\mathbb{Y}}\}.$$

The concept of Hausdorff well-posedness for the problem (PCUVOP) with respect to robust Benson efficient solutions is defined as follows.

Definition 5.4 The problem (PCUVOP) is said to be Hausdorff well-posed on Γ for robust Benson efficient solutions if for any $\gamma \in \Gamma$,

(i) $\mathrm{BEff}_K(\mathrm{PCUVOP})(\zeta, \eta, D)(\gamma) \neq \emptyset$;
(ii) the solution map $\widehat{\mathrm{BEff}}_K(\mathrm{PCUVOP})(\zeta, \eta, D)$ is upper Hausdorff semicontinuous at $(\gamma, 0)$.

We will conclude this section by providing sufficient conditions for the Hausdorff well-posedness of the problem (PCUVOP) for robust Benson efficient solutions. The proof of this result is approached similarly to that of Theorem 4.4, so we leave the details of the proof to the reader.

Theorem 5.2 *Assume that assumptions* $(\mathcal{A}0)$, $(\mathcal{A}'1)$, $(\mathcal{A}2)$, $(\mathcal{A}'3)$, $(\mathcal{A}4)$-$(\mathcal{A}6)$ *are fulfilled. Then, the problem* (PCUVOP) *is Hausdorff well-posed on* Γ.

Remark 5.1 (a) The models of robust optimization problems and parametric optimization problems share similarities in their construction. Both frameworks include structural elements where parameters (either uncertain or certain) influence the optimization process. However, their objectives differ. In robust optimization problems, due to the uncertainty of the scenario θ, the solution must account for all worst-case scenarios that the parameter may present. This often leads to conservative solution structures, as the problems focus on ensuring performance in the worst-case scenario. The solution not only needs to work well for a specific value of θ but must also remain effective across the entire set Θ, leading to solutions designed to withstand significant variations. In contrast, parametric optimization problems focus on finding the optimal solution for a specific value of the parameter p. In this case, the structure of the solution is tightly dependent on the parameter p, and any change in p typically requires recalculating the solution. Thus, the solution is local, depending on each specific value of p rather than the entire parameter space. Consequently, the solution conditions for robust optimization problems often involve stricter requirements related to the constraint set and objective map compared to those for parametric optimization problems. For instance, the qualitative properties of solutions in robust optimization problems frequently involve conditions such as uniform continuity or uniform convexity.

(b) From these observations, the main contributions of this section can be summarized as follows. First, the study reduces conservativeness in robust optimization. By achieving stability results for robust optimization problems under common assumptions used in parametric optimization problems, the study bridges the gap between the two frameworks, demonstrating that the stability of solutions in robust optimization problems can be achieved without overly stringent conditions. Second, the study extends the applicability of robust optimization problems. By reducing reliance on specialized assumptions, this approach makes robust optimization problems more accessible in scenarios typically addressed by parametric optimization problems. This broadens the potential applications of robust optimization problems to areas where precise information about uncertainty is not available, while still ensuring the stability of the model. Third, the study unifies the two frameworks. The results highlight the conceptual unification of robust and parametric optimization problems, showing that the uncertainty set in robust optimization problems can be viewed as a parameter space. The demonstration that stability in robust optimization models can be achieved under assumptions from parametric optimization models reveals that parametric optimization models, to some extent, serve as a natural generalization of robust optimization models. Finally, the theoretical significance of the study lies in achieving stability for robust optimization problems using assumptions from parametric optimization problems. This is not merely a technical convenience; it reflects the flexibility and generality of the stability framework. By enhancing the theoretical understanding of both optimization types, the study opens new research directions for cross-applications between them.

6 Conclusions

In this paper, we examined the qualitative properties of robust Benson efficient solutions for both unconstrained and constrained uncertain vector optimization problems involving free disposal sets. We established necessary and sufficient optimality conditions for robust Benson efficient solutions of these problems through the linear scalarization method. Additionally, we formulated the stability of robust Benson efficient solutions of the reference problems in terms of the semicontinuity properties of solution maps and Hausdorff well-posedness. We are confident that the techniques and methods proposed in this paper, with suitable adjustments, have the potential to be applied in exploring the qualitative properties of various types of efficient solutions for other uncertainty vector optimization models.

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Declarations

Conflict of interest The authors affirm that there are no actual or potential conflict of interest associated with this article.

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