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Stability Analysis to Parametric Multiobjective Optimal Control Problems

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Abstract

In this paper, we investigate continuity properties of the efficient solution map of a parametric nonlinear multiobjective optimal control problem. First, by using the equimeasurability condition of the admissible control set, we obtain the compactness and arcwise connectedness of the feasible solution set. Next, we suggest new concepts of the quasi-arcwise connected integrand and employ them to study the semicontinuity of the efficient solution map of this problem. When the multiobjective function does not satisfy these conditions, we propose an estimation hypothesis for approximate efficient solutions to address lower semicontinuity conditions of the efficient solution map of the reference problem. To illustrate the applicability, we apply the obtained results to two practical models, including Glucose model and Epidemic model.

Keywords Multiobjective optimal control · Glucose and epidemic models · Stability analysis · Arcwise connected integrand · Epi-convergence

Mathematics Subject Classification 49K40 · 90B50 · 93C15

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1 Introduction

Optimal control problems have a wide range of applications, spanning diverse fields, from space mission design and robotics to addressing economic challenges. The goal is to determine trajectories for both the system's state and control inputs that optimize an objective function. This function may involve minimizing energy consumption, maximizing comfort, or optimizing resource utilization. However, many real-world problems require us to optimize simultaneously not only one but multiple objectives that may be conflicting. Therefore, studies on multiobjective optimal control problems are important and significant due to the role of these models in various fields, for instance, economics, aerospace engineering, mechanical engineering, chemical engineering, multi-objective control system design, and environmental studies. We would like to refer the reader to the papers [4, 8, 10, 16, 22, 23, 26, 27, 33, 34, 41, 43] and the references therein for further comments and discussions. We would also like to note further that these works mainly devoted to optimality conditions and numerical methods for multiobjective optimal control problems.

The stability analysis for parametric optimal control problems is the next important topic, it has received much attention of many researchers recently, see e.g. [1, 2, 13, 17, 25, 28, 32, 39]. Unfortunately, because of the lack of techniques and tools, most of works only considered the stability of scalar optimal control problems. Let us provide a brief overview of recent stability results for such problems. In [17], the authors employed a strong second order optimality condition and uniform independence of active constraint gradients to establish the Lipschitz continuity of the solution map of a parametric nonlinear optimal control problem. When the strong second order optimality condition in [17] was unavailable, the author of [32] substituted it with constraint qualifications and weakened coercivity conditions to investigate the Lipschitz property and directional differential of the solution map and Lagrange multipliers for such problem. Motivated by [17, 32], in [2], the authors introduced a concept of convex integrand and utilized it together with Hölder continuity conditions to formulate Hölder conditions for a linear optimal control problem. Besides, by using the continuous differential and strong convexity conditions of the objective function, [25] studied the lower semicontinuity property of a parametric optimal control problem in the case where the state equation is linear and the cost function is convex in both variables. Subsequently, in [28], the authors developed the techniques presented in [25] to investigate the upper semicontinuity of the solution map of this problem, where the state equation is linear only in the control variable and the cost function is convex only in the control variable.

Moving to works dedicated to the differential stability for the reference problems, in [13], the authors derived a formula for an upper evaluation of the Fréchet subdifferential of the value function of a parametric optimal control problem by establishing an abstract result on the Fréchet subdifferential of the value function of a parametric minimization problem. Then, in [1, 39], the authors studied the first order behavior of the value function of a parametric convex optimal control problem. They used appropriate regularity conditions to obtain formulas for computing the subdifferential and the singular subdifferential of the optimal value function of the reference problem.

To the best of our knowledge, up to now, there are only two works devoted to the stability analysis for multiobjective optimal control problems. In [40], the authors first established an abstract result concerning the Mordukhovich subdifferential of the efficient point multifunction of a parametric multiobjective mathematical programming problem, and then it is applied to derive a formula for the computation of the Mordukhovich subdifferential of the efficient point multifunction of a multiobjective parametric optimal control problem. Recently, in [9], by employing Fréchet differential conditions, the authors considered the local Hölder continuity of the efficient solution map of a parametric multiobjective optimal control problem, where the state equation and multiobjective function are linear in the control variable. Therefore, a crucial topic such as stability analysis must be the focus of many works because the existing results are inadequate for its role. Moreover, according to our observations, the majority of works addressing stability conditions for optimal control problems under linear and nonlinear evolution equations have relied on conditions and tools concerning the continuous differential of objective functions as well as functions on the right-hand side of the evolution equations. However, in many practical situations as mentioned in [14, Sect. 1.1], these conditions are not met, and hence the study of new approaches and tools, especially those from multivalued analysis and convex analysis, is both deserved and significant.

Motivated by this research stream, in this paper our aim is to study continuity properties of the efficient solution map of a parametric nonlinear multiobjective optimal control problem without assuming any differential conditions. To be more precise, based on the equimeasurability and arcwise connectedness of the admissible control set, we investigate the compactness, arcwise connectedness and stability of the feasible solutions. For the stability of the efficient solutions, we first suggest concepts of the quasi-arcwise connected integrand related to the convex integrand introduced by Rockafellar [36], and then we use them together with the uniform continuity of maps to study the continuity of efficient solution maps of the reference problems. In the case where the multiobjective function does not satisfy the quasi-arcwise connected integrand, inspired by ideas of [24, 42], we propose an estimation condition of approximate solutions concerning a generalized form of the epi-convergence condition in Rockafellar and Wets [37]. After that, we employ it to obtain lower semicontinuity conditions for such problems. Moreover, we also have utilized a generalized boundedness condition of function on the right-hand side of the state equation, which is a unified form of the boundedness conditions introduced by Ekeland and Temam [20], Frankowska and Rampazzo [21] and Tammer [38], to get the boundedness of the trajectory set. Under this condition, we have relaxed the uniform continuity requirements for both the multiobjective function and the function on the right-hand side of the state equation, as well as the conditions for the admissible control set. As a result, we have obtained many new and significant results. Finally, as applications of obtained results, we consider two practical situations: Glucose and epidemic models [19, 30].

2 Preliminaries

We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^n$ the n-dimensional Euclidean space with the norm $|\cdot|$, by $\mathbb{R}^n_+$ the nonnegative orthant in $\mathbb{R}^n$. The Banach space $\mathcal{C}([t_0, t_1], \mathbb{R}^n)$ is the space of all continuous functions $x : [t_0, t_1] \to \mathbb{R}^n$ equipped with the norm $\|x\| = \max_{t\in[t_0,t_1]} |x(t)|$. By $\mathcal{L}^p([t_0, t_1], \mathbb{R}^n)$, $1 \le p \le \infty$, we denote the space of all the Lebesgue integrable functions defined on $[t_0, t_1]$ with the norm $\|\cdot\|_p$.

Let Ξ and Ω be nonempty closed subsets of $\mathbb{R}^l$ and $\mathbb{R}^r$, respectively, $\varphi : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^r \to \mathbb{R}^m$ and $\psi : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^r \to \mathbb{R}^n$ be given maps. We deal with the multiobjective optimal control problem of finding a control $u \in \mathcal{L}^p([t_0, t_1], \mathbb{R}^l)$ and a state $x \in \mathcal{C}([t_0, t_1], \mathbb{R}^n)$ which solve

$$\min \int_{t_0}^{t_1} \varphi\big(t, x(t), u(t), \lambda(t)\big)dt,$$

subject to

$$x(t) = x^0 + \int_{t_0}^{t} \psi\big(s, x(s), u(s), \lambda(s)\big)ds \tag{1}$$

and the control constraint

$$u(t) \in \Xi \quad \text{a.e. } \ t \in [t_0, t_1].$$

Here x^0 is a given vector in $\mathbb{R}^n$, λ is an element of the parameter space Λ which is defined by

$$\Lambda := \left\{\lambda \in \mathcal{L}^q([t_0, t_1], \mathbb{R}^r) : \lambda(t) \in \Omega \quad \forall t \in [t_0, t_1]\right\}.$$

Let Θ be a closed subset of $\mathbb{R}^n$ such that $x^0 \in \operatorname{int} \Theta$. An admissible control can be defined as a function $u \in \mathcal{L}^p([t_0, t_1], \mathbb{R}^l)$ satisfying the following conditions

(i) $u(t) \in \Xi$ a.e. $t \in [t_0, t_1]$;
(ii) for all $\lambda \in \Lambda$, the trajectory x corresponding to u satisfies $x(t) \in \Theta$ for all $t \in [t_0, t_1]$.

The set of all the admissible controls is denoted by $\mathbb{U}$.

Let $\mathcal{U}$ be a nonempty closed subset of $\mathbb{U}$, we define

$$\mathcal{X} := \mathcal{C}([t_0, t_1], \mathbb{R}^n),\ \mathcal{W} := \mathcal{X} \times \mathcal{U}.$$

For $\lambda \in \Lambda$, we define

$$\mathcal{I}(w, \lambda) := \int_{t_0}^{t_1} \varphi\big(t, x(t), u(t), \lambda(t)\big)dt, \tag{2}$$

$$K(\lambda) := \{w = (x, u) \in \mathcal{X} \times \mathcal{U} : (1) \text{ is satisfied}\}, \tag{3}$$

with $\mathcal{I} : \mathcal{W} \times \Lambda \to \mathbb{R}^m$ is a multiobjective function, and $K : \Lambda \rightrightarrows \mathcal{W}$ is a feasible solution map. Then, the multiobjective optimal control problem can be cast as the following problem

$$(\mathcal{P}(\lambda)): \quad \min \mathcal{I}(w, \lambda) \text{ subject to } w \in K(\lambda).$$

We denote the sets of all the minimal points and efficient solutions of $(\mathcal{P})$ at λ by $\mathrm{Min}(\lambda)$ and $\mathrm{Eff}(\lambda)$, respectively, i.e.,

$$\begin{aligned}\mathrm{Min}(\lambda) &:= \left\{a \in \mathcal{I}\big(K(\lambda), \lambda\big) : \big(\mathcal{I}(K(\lambda), \lambda) - a\big) \cap \big(-\mathbb{R}^m_+ \setminus \{\mathbf{0}\}\big) = \emptyset\right\},\\ \mathrm{Eff}(\lambda) &:= \{w \in K(\lambda) : \mathcal{I}(w, \lambda) \in \mathrm{Min}(\lambda)\}.\end{aligned}$$

Let $\lambda_0 \in \Lambda$ be a reference parameter. We call $(\mathcal{P}(\lambda_0))$ the original problem (or the unperturbed problem) and $(\mathcal{P}(\lambda))$ the perturbed problem. Our main concern is to investigate the behavior of $\mathrm{Eff}(\lambda)$ when λ varies around λ_0. For the details of the role, significance and need of considering parametric optimization models in general and optimal control models in particular, we refer the readers to typical works [7, 18, 35].

Definition 2.1 (see [5]) Let $\mathcal{M}_1, \mathcal{M}_2$ be metric spaces and $F : \mathcal{M}_1 \rightrightarrows \mathcal{M}_2$ be a set-valued map and $x_0 \in \mathcal{M}_1$. It is said that

(a) F is upper semicontinuous (usc, for short) at x_0 if for any neighborhood V of $F(x_0)$, there exists a neighborhood N of x_0 such that $F(N) \subset V$.
(b) F is lower semicontinuous (lsc, for short) at x_0 if for any sequence $\{x_n\}$ with $x_n \to x_0$ and $y_0 \in F(x_0)$, there exist $y_n \in F(x_n)$ such that $y_n \to y_0$.
(c) F is continuous at x_0 if it is both usc and lsc at x_0.

Lemma 2.1 (see [5]) *Let $F(x_0)$ be compact. Then, F is usc at x_0 if and only if for any sequence $\{x_n\}$ with $x_n \to x_0$ and $y_n \in F(x_n)$, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0 \in F(x_0)$.*

Definition 2.2 [31, p. 33, Definition 7.1] Let X, Y be normed spaces, $C \subset Y$ be a cone, and $f : X \to Y$ be a vector valued map. The map f is said to be upper C-semicontinuous at x_0 if for each neighborhood V of $f(x_0)$ in Y, there exists a neighborhood N of x_0 in X such that

$$f(x) \in V + C, \text{ for all } x \in N.$$

Lemma 2.2 [31, p.60, Corollary 5.10] *Let X, Y be normed spaces, $A \subset X$ be a nonempty subset, $C \subset Y$ be a pointed cone, and $f : X \to Y$ be a vector valued map. Suppose that A is compact, C is correct, i.e.* $\mathrm{cl}(C) + (C \setminus \{0_Y\}) \subset C$, *and f is upper C-semicontinuous. Then, the set*

$$\mathrm{Eff}(f, A) := \{x \in A : \big(f(A) - f(x)\big) \cap (-C \setminus \{0_Y\}) = \emptyset\}$$

is nonempty, where $\mathrm{cl}(C)$ *denotes the closure of C.*

Definition 2.3 (see [12]) Having a set Γ of maps γ defined on $[t_0, t_1]$, it is said that Γ is equimeasurable on $[t_0, t_1]$ if there is a sequence of step functions $\{\gamma_k\}$ for each $\gamma \in \Gamma$ satisfying

(i) for each k, γ_k has no more than k points of discontinuity,
(ii) for any $\varepsilon > 0$ and $\delta > 0$, there exists a finite number $k_0(\varepsilon, \delta)$ and for every γ_k with $k > k_0(\varepsilon, \delta)$, the inequality $|\gamma_k(t) - \gamma(t)| < \varepsilon$ holds except on a set of total measure less than δ.

Lemma 2.3 (see [12]) *Let Γ be an equimeasurable set of bounded functions. Then, for all $\{\gamma_k\} \subset \Gamma$, there exists a subsequence $\{\gamma_{k_i}\}$ of $\{\gamma_k\}$ such that $\{\gamma_{k_i}\}$ converges almost uniformly to a measurable function γ.*

Definition 2.4 (see [6]) Let $\mathcal{A}$ be a nonempty subset of $\mathcal{L}^p([t_0, t_1], \mathbb{R}^n)$. The set $\mathcal{A}$ is said to be arcwise connected if for all $w_1, w_2 \in \mathcal{A}$, there exists a continuous map $\xi : [0, 1] \to \mathcal{A}$ such that $\xi(0) = w_1$ and $\xi(1) = w_2$.

In what follows, we need the following hypotheses on $\varphi, \psi, \mathcal{U}$ and Ξ.

($\mathcal{H}1$) For a.e. $t \in [t_0, t_1]$, $\varphi(t, \cdot, \cdot, \cdot)$ is uniformly continuous on $\Theta \times \Xi \times \Omega$ and for each fixed $(y, v, \mu) \in \Theta \times \Xi \times \Omega$, $\varphi(\cdot, y, v, \mu)$ is measurable on $[t_0, t_1]$.

($\mathcal{H}2$) For a.e. $t \in [t_0, t_1]$ and for all $y \in \Theta$, $\psi(t, y, \cdot, \cdot)$ is uniformly continuous on $\Xi \times \Omega$ and for each fixed $(y, v, \mu) \in \Theta \times \Xi \times \Omega$, $\psi(\cdot, y, v, \mu)$ is measurable on $[t_0, t_1]$.

($\mathcal{H}3$) There exists a function $\ell \in \mathcal{L}^1([t_0, t_1], \mathbb{R})$ such that

$$|\psi(t, y_1, v, \mu) - \psi(t, y_2, v, \mu)| \leq \ell(t)|y_1 - y_2| \quad \text{a.e. } t \in [t_0, t_1], \forall (y_1, v, \mu), (y_2, v, \mu) \in \Theta \times \Xi \times \Omega.$$

($\mathcal{H}4$) $\mathcal{U}$ is equimeasurable on $[t_0, t_1]$ and Ξ is bounded.

The Hypotheses ($\mathcal{H}1$) $-$ ($\mathcal{H}3$) are very common in studying solution properties for optimal control models. In the remaining part of this section, we will discuss the equimeasurability condition in Hypothesis ($\mathcal{H}4$).

Assume that $\mathcal{U}$ is a set of piecewise equicontinuous maps on $[t_0, t_1]$, that is, there exists a positive integer number n_0 such that for each $u \in \mathcal{U}$, the number of discontinuous points of u is less than n_0, and for each $\epsilon > 0$, there exists $\sigma > 0$ such that for all $u \in \mathcal{U}$ and for all $t', t'' \in [t_0, t_1]$ with $|t' - t''| < \sigma$, we have

$$|u(t') - u(t'')| < \epsilon,$$

unless there is a point of discontinuity $\bar{t}$ of some function u with $t' < \bar{t} \leq t''$. According to Lemma 3.5 in [3], the set $\mathcal{U}$ is equimeasurable on $[t_0, t_1]$.

In the case that $\mathcal{U}$ is a set of maps with uniformly bounded variation on $[t_0, t_1]$, that is, there exists $\alpha_0 \in \mathbb{R}_+$ such that

$$\operatorname{Var}(u; [t_0, t_1]) := \sup\{\operatorname{Var}(u, P; [t_0, t_1]) : P \in \mathcal{P}([t_0, t_1])\} \leq \alpha_0 \text{ for all } u \in \mathcal{U},$$

where $\mathcal{P}([t_0, t_1])$ is a family of all partitions of the interval $[t_0, t_1]$, and $\operatorname{Var}(u, P; [t_0, t_1]) := \sum_{j=1}^{k} \left|u\left(\tau_j\right) - u\left(\tau_{j-1}\right)\right|$ with $P = \{\tau_0, \tau_1, \ldots, \tau_k\}$. Then, by Lemma 3.4 in [3], the set $\mathcal{U}$ is also equimeasurable on $[t_0, t_1]$.

3 Properties of the Feasible Solution Maps and Objective Maps

This section aims to study the compactness and arcwise connectedness of the feasible solution set and to investigate the continuity of both the feasible solution map and objective map. We start with the following lemma.

Lemma 3.1 *Assume that the Hypotheses* ($\mathcal{H}2$) *and* ($\mathcal{H}3$) *are fulfilled. Let* $\{\lambda_k\} \subset \Lambda$ *and* $(x_k, u_k) \in K(\lambda_k)$ *be arbitrary. If* $\{(u_k, \lambda_k)\}$ *converges to* $(u, \lambda) \in \mathcal{U} \times \Lambda$, *then the sequence* $\{x_k\}$ *converges to* $x \in \mathcal{X}$ *and* $(x, u) \in K(\lambda)$.

Proof For $u \in \mathcal{U}$ and $\lambda \in \Lambda$, by the Hypotheses ($\mathcal{H}2$) and ($\mathcal{H}3$), the Eq. (1) has a unique solution $x \in \mathcal{X}$, and hence $(x, u) \in K(\lambda)$. We now show that the sequence $\{x_k\}$ converges to x. For all $\varepsilon > 0$, because ψ is uniformly continuous in the third component and the fourth component on $\Xi \times \Omega$ and the sequence $\{(u_k, \lambda_k)\}$ converges to (u, λ), there exists $k_0 \in \mathbb{N}$ such that

$$|\psi\big(t, x(t), u_k(t), \lambda_k(t)\big) - \psi\big(t, x(t), u(t), \lambda(t)\big)| < \tfrac{\varepsilon}{(t_1 - t_0)} \quad a.e.\ t \in [t_0, t_1], \quad \forall k \geq k_0.$$

Then, for any $k \geq k_0$, due to $(x_k, u_k) \in K(\lambda_k)$ and the Hypothesis ($\mathcal{H}3$), we have

$$\begin{aligned} |x_k(t) - x(t)| &\leq \int_{t_0}^{t} |\psi(s, x_k(s), u_k(s), \lambda_k(s)) - \psi(s, x(s), u(s), \lambda(s))| ds \\ &\leq \int_{t_0}^{t} |\psi(s, x_k(s), u_k(s), \lambda_k(s)) - \psi(s, x(s), u_k(s), \lambda_k(s))| ds \\ &\quad + \int_{t_0}^{t} |\psi(s, x(s), u_k(s), \lambda_k(s)) - \psi(s, x(s), u(s), \lambda(s))| ds \\ &\leq \int_{t_0}^{t} \ell(s)|x_k(s) - x(s)| ds + \int_{t_0}^{t} \frac{\varepsilon}{(t_1 - t_0)} ds \\ &\leq \varepsilon + \int_{t_0}^{t} \ell(s)|x_k(s) - x(s)| ds, \quad t \in [t_0, t_1]. \end{aligned}$$

Applying the Gronwall lemma see [11, Lemma 18.1.i], for any $k \geq k_0$, we obtain

$$|x_k(t) - x(t)| \leq \varepsilon e^{\int_{t_0}^{t} \ell(s) ds} \leq \varepsilon e^{\|\ell\|_1} \quad \forall t \in [t_0, t_1].$$

Consequently, $\max_{t \in [t_0, t_1]} |x_k(t) - x(t)| \leq \varepsilon e^{\|\ell\|_1}$. Because $\varepsilon > 0$ is arbitrary, we conclude that $x_k \to x$. The proof is complete. □

Corollary 3.1 *Assume that the Hypotheses* ($\mathcal{H}2$) *and* ($\mathcal{H}3$) *are fulfilled. Let* $\lambda \in \Lambda$ *be fixed, and* $\{(x_k, u_k)\} \subset K(\lambda)$ *be arbitrary. If* $\{u_k\}$ *converges to* $u \in \mathcal{U}$, *then the sequence* $\{x_k\}$ *converges to* $x \in \mathcal{X}$ *and* $(x, u) \in K(\lambda)$.

Corollary 3.2 *Assume that the Hypotheses* ($\mathcal{H}2$) *and* ($\mathcal{H}3$) *are fulfilled. Let* $u \in \mathcal{U}$ *be fixed and* $\{\lambda_k\} \subset \Lambda$ *be arbitrary. If* $\{\lambda_k\}$ *converges to* $\lambda \in \Lambda$, *then the sequence* $\{x_k\}$ *with* $(x_k, u) \in K(\lambda_k)$ *converges to* $x \in \mathcal{X}$ *and* $(x, u) \in K(\lambda)$.

We are now in a position to present the main results of this section.

Theorem 3.1 *Assume that the Hypotheses* $(\mathcal{H}2) - (\mathcal{H}4)$ *are fulfilled. Then, for each* $\lambda \in \Lambda$, $K(\lambda)$ *is nonempty and compact. Moreover, if the set* $\mathcal{U}$ *is arcwise connected, then* $K(\lambda)$ *is arcwise connected.*

Proof For each $\lambda \in \Lambda$ and $u \in \mathcal{U}$, the Eq. (1) has a unique solution $x \in \mathcal{X}$, and hence the set $K(\lambda)$ is nonempty. Taking arbitrary $\{(x_k, u_k)\} \subset K(\lambda)$, then for the sequence $\{u_k\}$, by the Hypothesis $(\mathcal{H}4)$ and Lemma 2.3, there exists a subsequence $\{u_{k_i}\}$ converging to some u, and hence $u \in \mathcal{U}$ as $\mathcal{U}$ is closed. Applying Corollary 3.1, the subsequence $\{x_{k_i}\}$ converges to $x \in \mathcal{X}$ and $(x, u) \in K(\lambda)$. So, the subsequence $\{(x_{k_i}, u_{k_i})\}$ converges to $(x, u) \in K(\lambda)$, i.e., $K(\lambda)$ is compact.

Now, if in addition that the set $\mathcal{U}$ is arcwise connected, we will show that the feasible solution set $K(\lambda)$ is also arcwise connected. Taking arbitrary $w_0 = (x_0, u_0)$, $w_1 = (x_1, u_1) \in K(\lambda)$, then for u_0 and u_1 belonging to the arcwise connected set $\mathcal{U}$, there exists a continuous map $\xi_1 : [0, 1] \to \mathcal{U}$ satisfying $\xi_1(0) = u_0$, $\xi_1(1) = u_1$ and $\xi_1(\tau) = u_\tau \in \mathcal{U}$ for any $\tau \in (0, 1)$. On the other hand, for each $\tau \in [0, 1]$, thanks to the Hypotheses $(\mathcal{H}2)$ and $(\mathcal{H}3)$, the equation

$$x = x^0 + \int_{t_0}^{t} \psi(s, x(s), u_\tau(s), \lambda(s))ds \quad t \in [t_0, t_1]$$

has a unique solution $x_\tau \in \mathcal{X}$. Consider a map $\xi_2 : [0, 1] \to \mathcal{X}$ defined by $\xi_2(\tau) = x_\tau$. Taking arbitrary $\tau \in [0, 1]$ and a sequence $\{\tau_k\} \subset [0, 1]$ with $\tau_k \to \tau$, the continuity of ξ_1 on $[0, 1]$ leads to $u_{\tau_k} \to u_\tau \in \mathcal{U}$. Applying Corollary 3.1, we have $x_{\tau_k} \to x_\tau$, or equivalently $\xi_2(\tau_k) \to \xi_2(\tau)$ as $\tau_k \to \tau$. So, the map ξ_2 is continuous on $[0, 1]$. Setting $\xi := (\xi_2, \xi_1)$, then the map $\xi : [0, 1] \to K(\lambda)$ is continuous on $[0, 1]$, $\xi(0) = w_0$ and $\xi(1) = w_1$. $\square$

Theorem 3.2 *Assume that the Hypotheses* $(\mathcal{H}2) - (\mathcal{H}4)$ *are fulfilled. Then, the map* K *defined by (3) is continuous on* Λ.

Proof $\diamond$ *K is usc on Λ.*

For each $\lambda \in \Lambda$, it follows from Proposition 3.1 that $K(\lambda)$ is compact. Let $\{\lambda_k\} \subset \Lambda$ with $\lambda_k \to \lambda$ and $w_k = (x_k, u_k) \in K(\lambda_k)$ be arbitrary. By the Hypothesis $(\mathcal{H}4)$, there exists a subsequence $\{u_{k_i}\}$ of the sequence $\{u_k\}$ such that $\{u_{k_i}\}$ converges to some u. Due to the closedness of $\mathcal{U}$, we have $u \in \mathcal{U}$. Applying Lemma 3.1, we obtain $x_{k_i} \to x$ and $(x, u) \in K(\lambda)$. Therefore, $w_{k_i} = (x_{k_i}, u_{k_i}) \to w := (x, u) \in K(\lambda)$, and so, by Lemma 2.1, the map K is usc at λ.

$\diamond$ *K is lsc on Λ.*

For each $\lambda \in \Lambda$, we prove that K is lsc at λ. Let $\{\lambda_k\} \subset \Lambda$ with $\lambda_k \to \lambda$ and $w = (x, u) \in K(\lambda)$ be arbitrary, we show that there are vectors $w_k \in K(\lambda_k)$ such that $\{w_k\}$ converges to the vector w. For each $\lambda_k \in \Lambda$, the equations (1) has a unique solution x_k satisfying $x_k(t) = x_0 + \int_{t_0}^{t} \psi(s, x_k(s), u(s), \lambda_k(s))ds$. It follows from Corollary 3.2 that $x_k \to x$, and hence the sequence $\{w_k\}$ with $w_k := (x_k, u) \in K(\lambda_k)$ converges to the vector $w = (x, u)$. The proof is complete. $\square$

Theorem 3.3 *Assume that the Hypothesis* ($\mathcal{H}1$) *is fulfilled. Then, the map* $\mathcal{I}$ *is continuous on* $\mathcal{W} \times \Lambda$.

Proof Let $(w, \lambda) \in \mathcal{W} \times \Lambda$ and $\{(w_k, \lambda_k)\} \subset \mathcal{W} \times \Lambda$ be arbitrary with $(w_k, \lambda_k) \to (w, \lambda)$. For all $\varepsilon > 0$ and for a.e. $t \in [t_0, t_1]$, because the map $\varphi(t, \cdot, \cdot, \cdot)$ is uniformly continuous on $\Theta \times \Xi \times \Omega$, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\left|\varphi\big(t, x_k(t), u_k(t), \lambda_k(t)\big) - \varphi\big(t, x(t), u(t), \lambda(t)\big)\right| \\ &< \frac{\varepsilon}{(t_1 - t_0)} \quad a.e.\ t \in [t_0, t_1], \forall k \geq k_0.\end{aligned}$$

Then, for any $k \geq k_0$, we have

$$\begin{aligned}|\mathcal{I}(w_k, \lambda_k) - \mathcal{I}(w, \lambda)| &\leq \int_{t_0}^{t_1} |\varphi\big(t, x_k(t), u_k(t), \lambda_k(t)\big) - \varphi\big(t, x(t), u(t), \lambda(t)\big)|dt \\ &\leq \int_{t_0}^{t_1} \frac{\varepsilon}{(t_1 - t_0)} dt = \varepsilon.\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we conclude that $\mathcal{I}(w_k, \lambda_k) \to \mathcal{I}(w, \lambda)$. This completes the proof. □

4 Qualitative Properties of Efficient Solution Maps of ($\mathcal{P}$)

In this section, we establish sufficient conditions for the continuity of efficient solution maps of ($\mathcal{P}$). Motivated by Rockafellar [36] and Crespi et al. [15], we first propose concepts of generalized convexity integrand and arcwise connected integrand for a map, and then we use them to study the continuity of efficient solution maps. When such properties are not satisfied, we introduce a key hypothesis concerning epi-convergence and apply it to formulate the lower semicontinuity property of the efficient solution maps. We now introduce the concepts of the generalized convexity integrand.

Definition 4.1 Let $\mathcal{A}$ be a nonempty subset of $\mathcal{L}^r([t_0, t_1], \mathbb{R}^s)$. A map $f : [t_0, t_1] \times \mathbb{R}^s \to \mathbb{R}^m$ is said to be

(a) naturally $\mathbb{R}^m_+$-quasi-convex-like integrand with respect to $\mathcal{A}$ if for all $z_1, z_2 \in \mathcal{A}$, there exist $z_3 \in \mathcal{A}$ and $\tau \in [0, 1]$ such that

$$\int_{t_0}^{t_1} f(t, z_3(t))dt \in \tau \int_{t_0}^{t_1} f(t, z_1(t))dt + (1 - \tau) \int_{t_0}^{t_1} f(t, z_2(t))dt - \mathbb{R}^m_+.$$

(b) strictly natural $\mathbb{R}^m_+$-quasi-convex-like integrand with respect to $\mathcal{A}$ if for all $z_1, z_2 \in \mathcal{A}$ and $z_1 \neq z_2$, there exist $z_3 \in \mathcal{A}$ and $\tau \in [0, 1]$ such that

$$\int_{t_0}^{t_1} f(t, z_3(t))dt \in \tau \int_{t_0}^{t_1} f(t, z_1(t))dt + (1 - \tau) \int_{t_0}^{t_1} f(t, z_2(t))dt - \operatorname{int} \mathbb{R}^m_+.$$

We use the following hypothesis on φ to study the upper semicontinuity of efficient solution maps.

($\mathcal{H}5$) For each $\lambda \in \Lambda$ and for each $t \in [t_0, t_1]$, $\varphi(t, \cdot, \cdot, \lambda(t))$ is strictly natural $\mathbb{R}^m_+$-quasi-convex-like integrand with respect to $K(\lambda)$, that is, for all $w_1 = (x_1, u_1)$ and $w_2 = (x_2, u_2)$ belonging to $K(\lambda)$ with $w_1 \neq w_2$, there exist $w_3 = (x_3, u_3) \in K(\lambda)$ and $\tau \in [0, 1]$ such that

$$\int_{t_0}^{t_1} \varphi(t, x_3(t), u_3(t), \lambda(t))dt \in \tau \int_{t_0}^{t_1} \varphi(t, x_1(t), u_1(t), \lambda(t))dt + (1-\tau)\int_{t_0}^{t_1} \varphi(t, x_2(t), u_2(t), \lambda(t))dt - \operatorname{int} \mathbb{R}^m_+.$$

Theorem 4.1 *Assume that the Hypotheses* $(\mathcal{H}1)-(\mathcal{H}5)$ *are fulfilled. Then, the efficient solution map* Eff *is usc on* Λ.

Proof Let $\lambda_0 \in \Lambda$ be arbitrary. Suppose on the contrary that the map Eff is not usc at λ_0, then there exist an open set $\mathbb{V}$ containing $\mathrm{Eff}(\lambda_0)$, a sequence $\{\lambda_k\} \subset \Lambda$ converging to λ_0, and a sequence $\{w_k\}$ with $w_k = (x_k, u_k) \in \mathrm{Eff}(\lambda_k)$ such that $w_k \notin \mathbb{V}$ for all k. By the Hypothesis ($\mathcal{H}4$) and Lemma 2.3, the sequence $\{u_k\}$ has a subsequence $\{u_{k_i}\}$ converging almost uniformly to a measurable function u. The closedness of $\mathcal{U}$ leads to $u \in \mathcal{U}$. Applying Lemma 3.1, we obtain that the subsequence $\{x_{k_i}\}$ converges to some vector x and $(x, u) \in K(\lambda_0)$. Due to $w_k \notin \mathbb{V}$ for all k, we have $w = (x, u) \notin \mathrm{Eff}(\lambda_0)$. Hence, there exists $\bar{w} = (\bar{x}, \bar{u}) \in K(\lambda_0)$ such that

$$\mathcal{I}(\bar{w}, \lambda_0) - \mathcal{I}(w, \lambda_0) \in -\mathbb{R}^m_+ \setminus \{\mathbf{0}\}. \tag{4}$$

Since φ is strictly natural $\mathbb{R}^m_+$-quasi-convex-like with respect to $K(\lambda_0)$, there exist $\hat{w} = (\hat{x}, \hat{u}) \in K(\lambda_0)$ and $\tau \in [0, 1]$ such that

$$\int_{t_0}^{t_1} \varphi\big(t, \hat{x}(t), \hat{u}(t), \lambda_0(t)\big)dt \in \tau \int_{t_0}^{t_1} \varphi\big(t, \bar{x}(t), \bar{u}(t), \lambda_0(t)\big)dt + (1-\tau)\int_{t_0}^{t_1} \varphi\big(t, x(t), u(t), \lambda_0(t)\big)dt - \operatorname{int} \mathbb{R}^m_+.$$

The definition of $\mathcal{I}$ implies that $\mathcal{I}(\hat{w}, \lambda_0) \in \tau\mathcal{I}(\bar{w}, \lambda_0) + (1-\tau)\mathcal{I}(w, \lambda_0) - \operatorname{int} \mathbb{R}^m_+$. Combining this with (4), we have

$$\begin{aligned} \mathcal{I}(\hat{w}, \lambda_0) &\in \tau\big(\mathcal{I}(w, \lambda_0) - \mathbb{R}^m_+ \setminus \{\mathbf{0}\}\big) + (1-\tau)\mathcal{I}(w, \lambda_0) - \operatorname{int} \mathbb{R}^m_+ \\ &\subset \mathcal{I}(w, \lambda_0) - \operatorname{int} \mathbb{R}^m_+. \end{aligned} \tag{5}$$

By the lower semicontinuity of K at λ_0, for the element $\hat{w} \in K(\lambda_0)$, there exist vectors $\hat{w}_k \in K(\lambda_k)$ such that the sequence $\{\hat{w}_k\}$ converges to $\hat{w}$. Due to $w_{k_i} = (x_{k_i}, u_{k_i}) \in \mathrm{Eff}(\lambda_{k_i})$, one has

$$\mathcal{I}(\hat{w}_{k_i}, \lambda_{k_i}) - \mathcal{I}(w_{k_i}, \lambda_{k_i}) \notin -\mathbb{R}^m_+ \setminus \{\mathbf{0}\}.$$

The continuity of $\mathcal{I}$ implies that $\mathcal{I}(\hat{w}, \lambda_0) - \mathcal{I}(w, \lambda_0) \notin -\operatorname{int} \mathbb{R}^m_+$. This contradicts (5), and hence the proof follows. □

Convexity conditions are considered as crucial conditions in studying the stability for optimization models. In [15, 29], the authors used generalized convexity conditions, such as $\mathbb{R}^m_+$-quasi-convex and properly $\mathbb{R}^m_+$-quasi-convex, along with convergence forms of sequences of functions and sets, to investigate the Painlevé-Kuratowski convergence conditions for the sets of weak solutions, efficient solutions, and minimal points to vector optimization problems. In this research direction, we relax the concept of convexity into the concept of arcwise connected integrands and utilize them to study the stability in the sense of the lower semicontinuity of the solution map of problem $(\mathcal{P})$.

Definition 4.2 Let $\mathcal{A}$ be a nonempty arcwise connected subset of $\mathcal{L}^r([t_0, t_1], \mathbb{R}^s)$. A map $f : [t_0, t_1] \times \mathbb{R}^s \to \mathbb{R}^m$ is said to be $\mathbb{R}^m_+$-quasi-arcwise connected integrand with respect to $\mathcal{A}$ if for all $\vartheta \in \mathbb{R}^m$, and $z_1, z_2 \in \mathcal{A}$,

$$\int_{t_0}^{t_1} f\big(t, z_1(t)\big)dt \leq_{\mathbb{R}^m_+} \vartheta \text{ and } \int_{t_0}^{t_1} f\big(t, z_2(t)\big)dt \leq_{\mathbb{R}^m_+} \vartheta$$

imply the existence of a continuous map $\xi : [0, 1] \to \mathcal{A}$, $\xi(0) = z_1, \xi(1) = z_2$ such that $\int_{t_0}^{t_1} f(t, \xi(\tau)(t))dt \leq_{\mathbb{R}^m_+} \vartheta$ for all $\tau \in [0, 1]$.

We now give an example to illustrate Definition 4.2.

Example 4.1 Let $t_0 = 0, t_1 = 1, m = 2, \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined by

$$\mathcal{A}_1 = \left\{ z : z(t) = \begin{cases} at^2 + bt + a & \text{if } \ 0 \leq t < \dfrac{1}{2}, \\ 6b & \text{if } \ \dfrac{1}{2} \leq t \leq 1, \end{cases} a \in [0, 1], b \in [1, 2] \right\},$$

$$\mathcal{A}_2 = \left\{ z : z(t) = \begin{cases} at^3 + bt + a & \text{if } \ 0 \leq t < \dfrac{1}{2}, \\ 6b & \text{if } \ \dfrac{1}{2} \leq t \leq 1, \end{cases} a \in [0, 1], b \in [1, 2] \right\},$$

and let f be a piecewise $\mathbb{R}^2_+$-convex map from $[0, 1] \times \mathbb{R}$ into $\mathbb{R}^2$ defined by

$$f(t, y) = \begin{cases} f_1(t, y) & \text{if } \ 0 \leq y < 3, \\ f_2(t, y) & \text{if } \ y \geq 3, \end{cases}$$

where the maps $f_i = (f_i^1, f_i^2) : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}^2_+$ are continuous and $f_i^j : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}_+$ are convex such that for each $t \in [0, 1]$, $f_i^j(t, \cdot)$ are increase monotone on

$[0, 12]$ for all $i, j \in \{1, 2\}$. It is clear that the sets $\mathcal{A}_1$, $\mathcal{A}_2$ and

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \left\{ z : z(t) = \begin{cases} bt & \text{if } 0 \le t < \frac{1}{2}, \\ 6b & \text{if } \frac{1}{2} \le t \le 1, \end{cases} b \in [1, 2] \right\}$$

are convex, while the set $\mathcal{A}$ is not convex but it is arcwise connected. Moreover, for all $z \in \mathcal{A}_i$, we have $z(t) \in [0, 3)$ for all $t \in [0, \frac{1}{2})$ and $z(t) \in [3, 12]$ for all $t \in [\frac{1}{2}, 1]$. We will show that if f_i are $\mathbb{R}^2_+$-convex in the second component for all $i \in \{1, 2\}$, then f is $\mathbb{R}^2_+$-quasi-arcwise connected integrand with respect to $\mathcal{A}$.

Indeed, for all $\vartheta = (\vartheta^1, \vartheta^2) \in \mathbb{R}^2$, for all $z_1, z_2 \in \mathcal{A}$ such that

$$\int_0^1 f\big(t, z_1(t)\big)dt \le_{\mathbb{R}^2_+} \vartheta \text{ and } \int_0^1 f\big(t, z_2(t)\big)dt \le_{\mathbb{R}^2_+} \vartheta,$$

we need to show that there exists a continuous map $\xi : [0, 1] \to \mathcal{A}, \xi(0) = z_1, \xi(1) = z_2$ such that $\int_{t_0}^{t_1} f\big(t, \xi(\tau)(t)\big)dt \le_{\mathbb{R}^2_+} \vartheta$ for all $\tau \in [0, 1]$. We consider two distinct cases.

Case 1 $z_1, z_2 \in \mathcal{A}_i, i = 1, 2$.

Consider a continuous map $\xi : [0, 1] \to \mathcal{A}$ defined by $\xi(\tau) = \tau z_1 + (1 - \tau)z_2$, then for all $\tau \in [0, 1]$, we have $\xi(\tau) \in \mathcal{A}_i$ as $z_1, z_2 \in \mathcal{A}_i$, $i = 1, 2$, and we also have

$$0 \le \xi(\tau)(t) < 3, \ \forall t \in \left[0, \frac{1}{2}\right) \text{ and } 3 \le \xi(\tau)(t) \le 12, \ \forall t \in \left[\frac{1}{2}, 1\right].$$

Because f_i are $\mathbb{R}^2_+$-convex in the second component for all $i \in \{1, 2\}$, for all $\tau \in [0, 1]$, we have

$$\begin{aligned}
\int_0^1 f\big(t, \xi(\tau)(t)\big)dt &= \int_0^{\frac{1}{2}} f\big(t, \xi(\tau)(t)\big)dt + \int_{\frac{1}{2}}^1 f\big(t, \xi(\tau)(t)\big)dt \\
&= \int_0^{\frac{1}{2}} f_1\big(t, \xi(\tau)(t)\big)dt + \int_{\frac{1}{2}}^1 f_2\big(t, \xi(\tau)(t)\big)dt \\
&\le_{\mathbb{R}^2_+} \tau \int_0^{\frac{1}{2}} f_1\big(t, z_1(t)\big)dt + (1 - \tau) \int_0^{\frac{1}{2}} f_1\big(t, z_2(t)\big)dt + \\
&\quad + \tau \int_{\frac{1}{2}}^1 f_2\big(t, z_1(t)\big)dt + (1 - \tau) \int_{\frac{1}{2}}^1 f_2\big(t, z_2(t)\big)dt \\
&= \tau \int_0^1 f\big(t, z_1(t)\big)dt + (1 - \tau) \int_0^1 f\big(t, z_2(t)\big)dt \\
&\le_{\mathbb{R}^2_+} \tau\vartheta + (1 - \tau)\vartheta = \vartheta.
\end{aligned}$$

Case 2 $z_1 \in \mathcal{A}_i \setminus \mathcal{A}_j$ and $z_2 \in \mathcal{A}_j \setminus \mathcal{A}_i$, for $i, j \in \{1, 2\}$.

Without loss of generality, we can assume that $i = 1$, $j = 2$. Then, there exist real numbers $a_1, a_2 \in (0, 1]$ and $b_1, b_2 \in [1, 2]$ such that

$$z_1(t) = \begin{cases} a_1 t^2 + b_1 t + a_1 & \text{if} \quad 0 \leq t < \frac{1}{2}, \\ 6b_1 & \text{if} \quad \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$\text{and } z_2(t) = \begin{cases} a_2 t^3 + b_2 t + a_2 & \text{if} \quad 0 \leq t < \frac{1}{2}, \\ 6b_2 & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

Choosing $\bar{z}_1, \bar{z}_2 \in \mathcal{A}_1 \cap \mathcal{A}_2$ with

$$\bar{z}_1(t) = \begin{cases} b_1 t & \text{if} \quad 0 \leq t < \frac{1}{2}, \\ 6b_1 & \text{if} \quad \frac{1}{2} \leq t \leq 1, \end{cases} \text{ and } \bar{z}_2(t) = \begin{cases} b_2 t & \text{if} \quad 0 \leq t < \frac{1}{2}, \\ 6b_2 & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

Because $\bar{z}_i(t) \leq z_i(t)$ for all $t \in [0, 1]$ and f_i^j are increase monotone in the second component on $[0, 12]$ for $i, j \in \{1, 2\}$, we obtain

$$\begin{aligned}
\int_0^1 f\big(t, \bar{z}_1(t)\big)dt &= \int_0^{\frac{1}{2}} f_1\big(t, \bar{z}_1(t)\big)dt + \int_{\frac{1}{2}}^1 f_2\big(t, \bar{z}_1(t)\big)dt \\
&= \left(\int_0^{\frac{1}{2}} f_1^1\big(t, \bar{z}_1(t)\big)dt, \int_0^{\frac{1}{2}} f_1^2\big(t, \bar{z}_1(t)\big)dt\right) \\
&\quad + \left(\int_{\frac{1}{2}}^1 f_2^1\big(t, \bar{z}_1(t)\big)dt, \int_{\frac{1}{2}}^1 f_2^2\big(t, \bar{z}_1(t)\big)dt\right) \\
&\leq_{\mathbb{R}^2_+} \left(\int_0^{\frac{1}{2}} f_1^1\big(t, z_1(t)\big)dt, \int_0^{\frac{1}{2}} f_1^2\big(t, z_1(t)\big)dt\right) \\
&\quad + \left(\int_{\frac{1}{2}}^1 f_2^1\big(t, z_1(t)\big)dt, \int_{\frac{1}{2}}^1 f_2^2\big(t, z_1(t)\big)dt\right) \\
&= \int_0^{\frac{1}{2}} f_1\big(t, z_1(t)\big)dt + \int_{\frac{1}{2}}^1 f_2\big(t, z_1(t)\big)dt \\
&= \int_0^1 f\big(t, z_1(t)\big)dt \leq_{\mathbb{R}^2_+} \vartheta,
\end{aligned}$$

and

$$\int_0^1 f\big(t, \bar{z}_2(t)\big)dt = \int_0^{\frac{1}{2}} f_1\big(t, \bar{z}_2(t)\big)dt + \int_{\frac{1}{2}}^1 f_2\big(t, \bar{z}_2(t)\big)dt$$

$$\leq_{\mathbb{R}^2_+} \int_0^{\frac{1}{2}} f_1\big(t, z_2(t)\big)dt + \int_{\frac{1}{2}}^1 f_2\big(t, z_2(t)\big)dt$$
$$= \int_0^1 f\big(t, z_2(t)\big)dt \leq_{\mathbb{R}^2_+} \vartheta.$$

Consider a continuous map $\xi : [0, 1] \to \mathcal{A}$ defined by

$$\xi(\tau) = \begin{cases} (1-3\tau)z_1 + 3\tau\bar{z}_1 & \text{if } \ 0 \leq \tau < \dfrac{1}{3}, \\ (2-3\tau)\bar{z}_1 + (3\tau-1)\bar{z}_2 & \text{if } \ \dfrac{1}{3} \leq \tau < \dfrac{2}{3}, \\ (3-3\tau)\bar{z}_2 + (3\tau-2)z_2 & \text{if } \ \dfrac{2}{3} \leq \tau \leq 1. \end{cases}$$

Then, we can check that $\xi(0) = z_1, \xi(\frac{1}{3}) = \bar{z}_1, \xi(\frac{2}{3}) = \bar{z}_2, \xi(1) = z_2$,

$$\xi(\tau) \in \mathcal{A}_1, \ \forall \tau \in \Big(0, \frac{1}{3}\Big) \cup \Big(\frac{1}{3}, \frac{2}{3}\Big),$$

and

$$\xi(\tau) \in \mathcal{A}_2, \ \forall \tau \in \Big(\frac{2}{3}, 1\Big).$$

By direct computations, we obtain

$$\int_0^1 f(t, \xi(\tau)(t))dt \leq_{\mathbb{R}^2_+} \vartheta, \ \forall \tau \in [0, 1].$$

Therefore, f is $\mathbb{R}^2_+$-quasi-arcwise connected integrand with respect to $\mathcal{A}$.

To exploit the $\mathbb{R}^m_+$-quasi-arcwise connected integrand property defined in Definition 4.2, we investigate the upper semicontinuity in the sense of Hausdorff for the level set of the multiobjective function of the problem $(\mathcal{P})$, detailed as follows.

For $\vartheta \in \mathbb{R}^m$, we consider the following lower level set

$$\mathrm{Lev}_{\leq \vartheta}\big(\mathcal{I}, \lambda\big) := \Big\{ w \in K(\lambda) : \mathcal{I}(w, \lambda) \leq_{\mathbb{R}^m_+} \vartheta \Big\}.$$

Lemma 4.1 *Let $\lambda_k, \lambda_0 \in \Lambda$ with $\lambda_k \to \lambda_0$ and $\vartheta_k, \vartheta_0 \in \mathbb{R}^m$ with $\vartheta_k \to \vartheta_0$. Assume that the Hypotheses $(\mathcal{H}1) - (\mathcal{H}4)$ are fulfilled and assume further that*

(i) $\mathcal{U}$ is arcwise connected, and for each $t \in [t_0, t_1]$, $\varphi(t, \cdot, \cdot, \lambda_k(t))$ is $\mathbb{R}^m_+$-quasi-arcwise connected integrand with respect to $K(\lambda_k)$ for each $k \in \mathbb{N}$;
(ii) $\mathrm{Lev}_{\leq \vartheta_0}\big(\mathcal{I}, \lambda_0\big)$ is nonempty.

Then, for all $\varepsilon > 0$*, it holds that*

$$\mathrm{Lev}_{\leq \vartheta_k}\left(\mathcal{I}, \lambda_k\right) \subset \mathrm{Lev}_{\leq \vartheta_0}\left(\mathcal{I}, \lambda_0\right) + \varepsilon \overline{\mathbb{B}}_{\mathcal{W}}, \tag{6}$$

for k *large enough, where* $\overline{\mathbb{B}}_{\mathcal{W}}$ *stands for the closed unit ball in the space* $\mathcal{W}$*.*

Proof Suppose on the contrary that one can find a real number $\varepsilon_0 > 0$ and a subsequence $\{\vartheta_{k_i}\}$ of the sequence $\{\vartheta_k\}$ such that for each k_i, there exists a point $w_{k_i} \in \mathrm{Lev}_{\leq \vartheta_{k_i}}\left(\mathcal{I}, \lambda_{k_i}\right)$ but

$$w_{k_i} \notin \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon_0 \overline{\mathbb{B}}_{\mathcal{W}}.$$

Then, for a fixed vector $e \in \mathrm{int}\, \mathbb{R}^m_+$ and for any real number $\alpha > 0$, due to $\vartheta_{k_i} \to \vartheta_0$, we have $\vartheta_{k_i} \in \vartheta_0 - \mathbb{R}^m_+ + \alpha e$ for i large enough. This together with the fact that $w_{k_i} \in \mathrm{Lev}_{\leq \vartheta_{k_i}}\left(\mathcal{I}, \lambda_{k_i}\right)$ yields that $\mathcal{I}(w_{k_i}, \lambda_{k_i}) \in \vartheta_0 - \mathbb{R}^m_+ + \alpha e$ for i large enough, i.e.,

$$w_{k_i} \in \mathrm{Lev}_{\leq \vartheta_0 + \alpha e}\left(\mathcal{I}, \lambda_{k_i}\right). \tag{7}$$

Now, taking arbitrary $\bar{w} \in \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0)$, then $\bar{w} \in K(\lambda_0)$ and

$$\mathcal{I}(\bar{w}, \lambda_0) \in \vartheta_0 - \mathbb{R}^m_+. \tag{8}$$

From the lower semicontinuity of K at λ_0, there exists $\bar{w}_{k_i} \in K(\lambda_{k_i})$ such that $\{\bar{w}_{k_i}\}$ converges to $\bar{w}$. By Theorem 3.3, we have $\mathcal{I}(\bar{w}_{k_i}, \lambda_{k_i}) \to \mathcal{I}(\bar{w}, \lambda_0)$. Because $\alpha > 0$ and $e \in \mathrm{int}\, \mathbb{R}^m_+$, (8) implies that the set $\vartheta_0 - \mathbb{R}^m_+ + \alpha e$ is a neighborhood of $\mathcal{I}(\bar{w}, \lambda_0)$. Therefore, $\mathcal{I}(\bar{w}_{k_i}, \lambda_{k_i}) \in \vartheta_0 - \mathbb{R}^m_+ + \alpha e$ for i large enough, i.e.,

$$\bar{w}_{k_i} \in \mathrm{Lev}_{\leq \vartheta_0 + \alpha e}\left(\mathcal{I}, \lambda_{k_i}\right). \tag{9}$$

For each i, because $\mathcal{U}$ is arcwise connected, by Theorem 3.1, $K(\lambda_{k_i})$ is arcwise connected. Then, from (7), (9) and Condition (i), for each i large enough, there exists a continuous map $\xi_{k_i} : [0, 1] \to K(\lambda_{k_i})$ with $\xi_{k_i}(0) = w_{k_i}$, $\xi_{k_i}(1) = \bar{w}_{k_i}$ such that $\xi_{k_i}(\tau) \in \mathrm{Lev}_{\leq \vartheta_0 + \alpha e}\left(\mathcal{I}, \lambda_{k_i}\right)$ for every $\tau \in [0, 1]$, i.e., $\mathcal{I}(\xi_{k_i}(\tau), \lambda_{k_i}) \in \vartheta_0 - \mathbb{R}^m_+ + \alpha e$ for every $\tau \in [0, 1]$. We now show that, for each i large enough, there exists a number $\tau_{k_i} \in [0, 1]$ such that

$$\xi_{k_i}(\tau_{k_i}) \in \mathrm{bd}\left(\mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon_0 \overline{\mathbb{B}}_{\mathcal{W}}\right), \tag{10}$$

where $\mathrm{bd}(E)$ stands for the boundary of E. Indeed, as $\bar{w}_{k_i} \to \bar{w}$ and $\bar{w} \in \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0)$, one has $\bar{w}_{k_i} \in \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon_0 \overline{\mathbb{B}}_{\mathcal{W}}$, for i large enough, while $w_{k_i} \notin \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon_0 \overline{\mathbb{B}}_{\mathcal{W}}$ due to the contrary assumption. Because of the continuity of ξ_{k_i} and the compactness of $\mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon_0 \overline{\mathbb{B}}_{\mathcal{W}}$, the intersection of $\xi_{k_i}([0, 1])$ and $\mathrm{bd}\left(\mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon_0 \overline{\mathbb{B}}_{\mathcal{W}}\right)$ must be nonempty, and so we obtain (10). Setting

$\hat{w}_{k_i} = (\hat{x}_{k_i}, \hat{u}_{k_i}) := \xi_{k_i}(\tau_{k_i})$. Taking into account $(\mathcal{H}4)$, by Lemma 2.3, the sequence $\{\hat{u}_{k_i}\}$ has a subsequence (still denoted $\{\hat{u}_{k_i}\}$ for simplification) converging to some $\hat{u} \in \mathcal{U}$. Applying Lemma 3.1, one has $\hat{x}_{k_i} \to \hat{x}$ and $(\hat{x}, \hat{u}) \in K(\lambda_0)$. Therefore, we have

$$\hat{w}_{k_i} \to \hat{w} = (\hat{x}, \hat{u}) \in \operatorname{bd}\left(\operatorname{Lev}_{\leq\vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon_0 \overline{\mathbb{B}}_{\mathcal{W}}\right). \tag{11}$$

Theorem 3.3 leads to $\mathcal{I}(\hat{w}_{k_i}, \lambda_{k_i}) \to \mathcal{I}(\hat{w}, \lambda_0)$ and $\mathcal{I}(\hat{w}, \lambda_0) \in \vartheta_0 - \mathbb{R}^m_+ + \alpha e$ as $\mathcal{I}(\hat{w}_{k_i}, \lambda_{k_i}) \in \vartheta_0 - \mathbb{R}^m_+ + \alpha e$. Since α is arbitrary, we conclude $\mathcal{I}(\hat{w}, \lambda_0) \in \vartheta_0 - \mathbb{R}^m_+$, i.e., $\hat{w} \in \operatorname{Lev}_{\leq\vartheta_0}(\mathcal{I}, \lambda_0)$, which contradicts (11). The proof follows. □

We need the following Hypothesis on φ to study the lower semicontinuity of efficient solution maps.

$(\mathcal{H}6)$ $\mathcal{U}$ is arcwise connected, and for each $\lambda_0 \in \Lambda$, there exists a neighborhood N of λ_0 such that for all $\lambda \in N$ and for each $t \in [t_0, t_1]$, $\varphi(t, \cdot, \cdot, \lambda(t))$ is $\mathbb{R}^m_+$-quasi-arcwise connected integrand with respect to $K(\lambda)$.

In [15], the authors employed the condition of strict quasi-convexity at the reference point λ_0, which is stronger than the condition of quasi-arcwise connected integrand defined in Definition 4.2, to study convergence criteria for the sets of efficient solutions and minimal points of vector optimization problems. Also, they showed that under this strict quasi-convexity condition, the efficient solution set at λ_0 is not a singleton in general, and the inverse image set of each minimal point contains a unique element.

Theorem 4.2 *Assume that the Hypotheses* $(\mathcal{H}1)-(\mathcal{H}6)$ *are fulfilled. Then, the efficient solution map* Eff *is lsc on* Λ.

Proof For any $\lambda \in \Lambda$, by the Hypotheses $(\mathcal{H}2)-(\mathcal{H}4)$, Theorem 3.1 implies that $K(\lambda)$ is compact. Thank to the Hypothesis $(\mathcal{H}1)$ and Theorem 3.3, we obtain the continuity of the multiobjective map $\mathcal{I}$. Then, by applying Lemma 2.2, we conclude that Eff is nonempty valued on Λ. Taking arbitrary $\lambda_0 \in \Lambda$ and $w_0 = (x_0, u_0) \in \operatorname{Eff}(\lambda_0)$. For any sequence $\{\lambda_k\} \subset \Lambda$ with $\lambda_k \to \lambda_0$, we prove that there exists a sequence $\{w_k\}$ with $w_k \in \operatorname{Eff}(\lambda_k)$ such that $\{w_k\}$ converges to w_0. Let $\vartheta_0 = \mathcal{I}(w_0, \lambda_0) \in \operatorname{Min}(\lambda_0)$ and consider the lower level set $\operatorname{Lev}_{\leq\vartheta_0}(\mathcal{I}, \lambda_0)$. We now show that

$$\operatorname{Lev}_{\leq\vartheta_0}(\mathcal{I}, \lambda_0) = \{w_0\}. \tag{12}$$

Suppose on the contrary that there is $w \in \operatorname{Lev}_{\leq\vartheta_0}(\mathcal{I}, \lambda_0)$ such that $w \neq w_0$. By Hypothesis $(\mathcal{H}5)$, there exists a vector $\bar{w} = (\bar{x}, \bar{u}) \in K(\lambda_0)$ and $\tau \in [0, 1]$ such that

$$\int_{t_0}^{t_1} \varphi(t, \bar{x}(t), \bar{u}(t), \lambda_0(t))dt \in \tau \int_{t_0}^{t_1} \varphi(t, x_0(t), u_0(t), \lambda_0(t))dt +$$
$$+ (1-\tau) \int_{t_0}^{t_1} \varphi(t, x(t), u(t), \lambda_0(t))dt - \operatorname{int} \mathbb{R}^m_+.$$

Combining this with the definition of $\mathcal{I}$ and the fact that $w_0, w \in \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0)$, we have the following evaluations

$$\mathcal{I}(\bar{w}, \lambda_0) <_{\mathbb{R}^m_+} \tau \mathcal{I}(w_0, \lambda_0) + (1-\tau)\mathcal{I}(w, \lambda_0) \leq_{\mathbb{R}^m_+} \tau \vartheta_0 + (1-\tau)\vartheta_0 = \vartheta_0.$$

This is impossible as $\vartheta_0 \in \mathrm{Min}(\lambda_0)$. So, we achieve the conclusion (12). On the other hand, for the vector $w_0 \in K(\lambda_0)$, by the lower semicontinuity of K at λ_0, there exists a sequence $\{\hat{w}_k\}$ with $\hat{w}_k \in K(\lambda_k)$ such that the sequence $\{\hat{w}_k\}$ converges to w_0. Applying Theorem 3.3, we get $\mathcal{I}(\hat{w}_k, \lambda_k) \to \mathcal{I}(w_0, \lambda_0) = \vartheta_0$, and hence for a fixed vector $e \in \mathrm{int}\, \mathbb{R}^m_+$, we can find a sequence $\{\alpha_k\}$ with $\alpha_k \searrow 0$ such that $\mathcal{I}(\hat{w}_k, \lambda_k) \in \vartheta_0 + \alpha_k e - \mathbb{R}^m_+$ for k large enough, i.e., $\hat{w}_k \in \mathrm{Lev}_{\leq \vartheta_0 + \alpha_k e}(\mathcal{I}, \lambda_k)$ for k large enough. Using Lemma 4.1, for all $\varepsilon > 0$, for k large enough, one has

$$\mathrm{Lev}_{\leq \vartheta_0 + \alpha_k e}(\mathcal{I}, \lambda_k) \subset \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon \overline{\mathbb{B}}_{\mathcal{W}}. \tag{13}$$

For each k large enough, we consider the auxiliary problem as follows

$$(\widehat{\mathcal{P}}_k): \quad \min_{\mathbb{R}^m_+} \mathcal{I}(w, \lambda_k) \text{ subject to } w \in \mathrm{Lev}_{\leq \vartheta_0 + \alpha_k e}(\mathcal{I}, \lambda_k).$$

Denote by $\widehat{\mathrm{Eff}}(\lambda_k)$ the set of all efficient solutions of $(\widehat{\mathcal{P}}_k)$. By the Hypotheses $(\mathcal{H}1)-(\mathcal{H}4)$, the map $\mathcal{I}$ is continuous and the set $K(\lambda_k)$ is compact. Consequently, we have $\mathrm{Lev}_{\leq \vartheta_0 + \alpha_k e}(\mathcal{I}, \lambda_k)$ is nonempty and compact, and hence $\widehat{\mathrm{Eff}}(\lambda_k) \neq \emptyset$ for each k. Choosing $w_k \in \widehat{\mathrm{Eff}}(\lambda_k)$, then we also have

$$w_k \in \mathrm{Lev}_{\leq \vartheta_0 + \alpha_k e}(\mathcal{I}, \lambda_k). \tag{14}$$

If $w_k \notin \mathrm{Eff}(\lambda_k)$, there exists $\tilde{w}_k \in K(\lambda_k)$ such that

$$\mathcal{I}(\tilde{w}_k, \lambda_k) \in \mathcal{I}(w_k, \lambda_k) - \mathbb{R}^m_+ \setminus \{\mathbf{0}\}. \tag{15}$$

On the other hand, by (14), one has

$$\mathcal{I}(w_k, \lambda_k) \in \vartheta_0 + \alpha_k e - \mathbb{R}^m_+. \tag{16}$$

Combining (15) and (16), we obtain

$$\mathcal{I}(\tilde{w}_k, \lambda_k) \in \vartheta_0 + \alpha_k e - \mathbb{R}^m_+ - \mathbb{R}^m_+ \setminus \{\mathbf{0}\} \subset \vartheta_0 + \alpha_k e - \mathbb{R}^m_+.$$

It follows that $\tilde{w}_k \in \mathrm{Lev}_{\leq \vartheta_0 + \alpha_k e}(\mathcal{I}, \lambda_k)$, and hence (15) implies that $w_k \notin \widehat{\mathrm{Eff}}(\lambda_k)$, which is a contradiction. Therefore, we obtain $w_k \in \mathrm{Eff}(\lambda_k)$. Moreover, by combining (12), (13) and (14), we have, for all $\varepsilon > 0$,

$$w_k \in \mathrm{Lev}_{\leq \vartheta_0}(\mathcal{I}, \lambda_0) + \varepsilon \overline{\mathbb{B}}_{\mathcal{W}} = w_0 + \varepsilon \overline{\mathbb{B}}_{\mathcal{W}},$$

for k large enough, or equivalently $w_k \to w_0$. The proof is complete. □

In the case that the map φ does not satisfy the above quasi-arcwise connected integrand properties, we introduce a key hypothesis concerning epi-convergence to study the lower semicontinuity of the efficient solution map. Motivated by Rockafellar and Wets [37], we recall the following definition and characterization of epi-convergence.

Definition 4.3 [37, Definition 7.1] Let $f, f_k : \mathbb{R}^n \to \mathbb{R}$ be given functions. The sequence $\{f_k\}$ is said to epi-converge to f, denoted by $f_k \stackrel{epi}{\to} f$, if

$$\liminf (\text{epi } f_k) = \limsup (\text{epi } f_k) = \text{epi } f.$$

Lemma 4.2 [37, Proposition 7.2] *Let $f, f_k : \mathbb{R}^n \to \mathbb{R}$ be given functions. Then, $f_k \stackrel{epi}{\to} f$ if and only if at each point $\vartheta \in \mathbb{R}^n$ one has*

$$\begin{cases} \liminf f_k(\vartheta_k) \geq f(\vartheta) & \text{for every sequence } \vartheta_k \to \vartheta, \\ \limsup f_k(\vartheta_k) \leq f(\vartheta) & \text{for some sequence } \vartheta_k \to \vartheta. \end{cases}$$

Motivated by the above lemma, we suggest the following generalized epi-convergence concept.

Definition 4.4 Let $f, f_k : \mathbb{R}^n \to \mathbb{R}$ be given functions. The sequence $\{f_k\}$ is said to *pseudo epi-converge* to f at ϑ, denoted by $f_k \xrightarrow{pepi} f$, if there exists a sequence $\{\vartheta_k\}$ converging to ϑ such that

$$\limsup f_k(\vartheta_k) \leq f(\vartheta).$$

Let $e \in \text{int}\, \mathbb{R}^m_+$ be a given vector, and let $\widetilde{\text{Eff}} : \Lambda \times \mathbb{R}_+ \rightrightarrows \mathcal{W}$ be the set-valued maps defined as

$$\widetilde{\text{Eff}}(\lambda, \varepsilon) := \Big\{ w \in K(\lambda) : \Big(\mathcal{I}(K(\lambda), \lambda) + \varepsilon e - \mathcal{I}(w, \lambda) \Big) \cap \Big(-\mathbb{R}^m_+ \setminus \{\mathbf{0}\} \Big) = \emptyset \Big\}.$$

It is called the ε-solution maps and we have $\widetilde{\text{Eff}}(\lambda, 0) = \text{Eff}(\lambda)$. We now propose the key hypothesis as follows.

($\mathcal{H}7$): For a given element $\lambda \in \Lambda$ and for each sequence $\{\lambda_k\} \subset \Lambda$ converging to λ, there exist functions $h, h_k : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(0) = 0$ and $h_k \xrightarrow{pepi} h$ at 0 such that for each $\{\varepsilon_k\} \subset \mathbb{R}_+$ converging to 0, then for k large enough,

$$d\big(w_k, \text{Eff}(\lambda_k)\big) \leq h_k(\varepsilon_k), \ \ \forall w_k \in \widetilde{\text{Eff}}(\lambda_k, \varepsilon_k). \tag{17}$$

In the geometrical sense, the hypothesis ($\mathcal{H}7$) means that the excess of the approximate efficient solution set beyond the efficient solution set is bounded from above by the function of the approximate variable having the limsup equal to 0. In the analytical sense, this hypothesis is closely related to the upper Hausdorff semicontinuity property of the approximate efficient solution map as (17) can be rewritten in the form of

$$\widetilde{\text{Eff}}(\lambda_k, \varepsilon_k) \subset \widetilde{\text{Eff}}(\lambda_k, 0) + h_k(\varepsilon_k)\mathbb{B}_{\mathcal{W}}.$$

Now, we employ the Hypotheses $(\mathcal{H}1) - (\mathcal{H}4)$ to formulate a property closely related to the lower semicontinuity of the approximate efficient solution map of the problem $(\mathcal{P})$.

Lemma 4.3 *For each $\lambda \in \Lambda$, and for each $\{\lambda_k\} \subset \Lambda$ converging to λ, assume that the Hypotheses $(\mathcal{H}1) - (\mathcal{H}4)$ are fulfilled. Then, for any $w \in \mathrm{Eff}(\lambda)$, there exists a sequence $\{w_k\}$ with $w_k \in K(\lambda_k)$ such that $w_k \to w$ and furthermore*

$$\forall \varepsilon > 0, \exists \bar{k}(\varepsilon) \in \mathbb{N}, \forall k \geq \bar{k}(\varepsilon), w_k \in \widetilde{\mathrm{Eff}}(\lambda_k, \varepsilon). \tag{18}$$

Proof For $w \in \mathrm{Eff}(\lambda) \subset K(\lambda)$, by the lower semicontinuity of K at λ, there exists a sequence $\{w_k\}$, $w_k \in K(\lambda_k)$, converging to w. We now prove (18). Suppose on the contrary that there exists $\varepsilon_0 > 0$ such that for all $j \in \mathbb{N}$, there is $k_j \geq j$ satisfying $w_{k_j} \notin \widetilde{\mathrm{Eff}}(\lambda_{k_j}, \varepsilon_0)$. Then, for each $j \in \mathbb{N}$, there exists $\bar{w}_{k_j} = (\bar{x}_{k_j}, \bar{u}_{k_j}) \in K(\lambda_{k_j})$ such that

$$\mathcal{I}\left(\bar{w}_{k_j}, \lambda_{k_j}\right) - \mathcal{I}\left(w_{k_j}, \lambda_{k_j}\right) + \varepsilon_0 e \in -\mathbb{R}^m_+ \setminus \{\mathbf{0}\}. \tag{19}$$

From the Hypothesis $(\mathcal{H}4)$, using Lemma 2.3, we can assume that the subsequence $\{\bar{u}_{k_j}\}$ converges to $\bar{u} \in \mathcal{U}$. Applying Lemma 3.1, we obtain $\bar{w}_{k_j} \to \bar{w} = (\bar{x}, \bar{u}) \in K(\lambda)$. It follows from (19) and the continuity of $\mathcal{I}$ that

$$\mathcal{I}(\bar{w}, \lambda_0) - \mathcal{I}(w, \lambda) + \varepsilon_0 e \in -\mathbb{R}^m_+,$$

or equivalently $\mathcal{I}(\bar{w}, \lambda_0) - \mathcal{I}(w, \lambda) \in -\varepsilon_0 e - \mathbb{R}^m_+$. Combining this with the fact that $-\varepsilon_0 e \in -\operatorname{int} \mathbb{R}^m_+$, we have

$$\mathcal{I}(\bar{w}, \lambda) - \mathcal{I}(w, \lambda) \in -\operatorname{int} \mathbb{R}^m_+ \subset -\mathbb{R}^m_+ \setminus \{\mathbf{0}\}.$$

This is impossible as $w \in \mathrm{Eff}(\lambda)$, and hence (18) holds. □

Theorem 4.3 *Assume that the Hypotheses $(\mathcal{H}1) - (\mathcal{H}4)$ are fulfilled. Then, the efficient solution map* Eff *is lower semicontinuous on Λ if the Hypothesis $(\mathcal{H}7)$ is satisfied on Λ.*

Proof Taking arbitrary $\lambda_0 \in \Lambda$, $w_0 = (x_0, u_0) \in \mathrm{Eff}(\lambda_0)$ and $\{\lambda_k\} \subset \Lambda$ converging to λ_0, we will show that there exist vectors $w_k \in \mathrm{Eff}(\lambda_k)$ such that the sequence $\{w_k\}$ converges to w_0. By the Hypothesis $(\mathcal{H}7)$, there exist functions $h, h_k : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(0) = 0$ and $h_k \xrightarrow{pepi} h$ at 0 such that for all $\{\varepsilon_k\} \subset \mathbb{R}_+$ converging to 0, then

$$d\big(\tilde{w}_k, \mathrm{Eff}(\lambda_k)\big) \leq h_k(\varepsilon_k), \ \forall \tilde{w}_k \in \widetilde{\mathrm{Eff}}(\lambda_k, \varepsilon_k), \tag{20}$$

for k large enough. Since $h_k \xrightarrow{pepi} h$ at 0, there exists a sequence $\{\hat{\epsilon}_k\} \subset \mathbb{R}_+$ converging to 0 such that

$$\limsup h_k(\hat{\epsilon}_k) \leq h(0) = 0.$$

For the vector $w_0 \in \mathrm{Eff}(\lambda_0)$ and the sequence $\{\hat{\epsilon}_i\}$, applying Lemma 4.3, there exists a sequence $\{\hat{w}_k\}$ with $\hat{w}_k \in K(\lambda_k)$ such that $\hat{w}_k \to w_0$ and for each i, one has

$$\exists \hat{k}(\hat{\epsilon}_i) \in \mathbb{N}, \forall k \geq \hat{k}(\hat{\epsilon}_i), \hat{w}_k \in \widetilde{\mathrm{Eff}}(\lambda_k, \hat{\epsilon}_i), \tag{21}$$

where the map $i \mapsto \hat{k}(\hat{\epsilon}_i)$ can be assumed to be increasing. For every $k \geq \hat{k}(\hat{\epsilon}_1)$, there exists a unique $i(k) \in \mathbb{N}$ such that

$$\hat{k}(\hat{\epsilon}_{i(k)}) \leq k < \hat{k}(\hat{\epsilon}_{i(k)+1}). \tag{22}$$

By setting $\bar{\epsilon}_k := \hat{\epsilon}_{i(k)}$ for $k = \hat{k}(\hat{\epsilon}_1), \hat{k}(\hat{\epsilon}_1) + 1, \ldots$, and it is obvious that the sequence $\{\bar{\epsilon}_k\}$ converges to 0 and

$$\limsup h_k(\bar{\epsilon}_k) \leq 0. \tag{23}$$

Then, for every $k \geq \hat{k}(\hat{\epsilon}_1)$ satisfying (22), we have $k \geq \hat{k}(\bar{\epsilon}_k)$. Therefore, from (21), we obtain $\hat{w}_k \in \widetilde{\mathrm{Eff}}(\lambda_k, \bar{\epsilon}_k)$. Combining this with (20), we have

$$d\big(\hat{w}_k, \mathrm{Eff}(\lambda_k)\big) \leq h_k(\bar{\epsilon}_k),$$

or equivalently $\inf_{z_k \in \mathrm{Eff}(\lambda_k)} d(\hat{w}_k, z_k) \leq h_k(\bar{\epsilon}_k)$. From the definition of infimum, for each $k \geq \hat{k}(\hat{\epsilon}_1)$, there exists $w_k \in \mathrm{Eff}(\lambda_k)$ such that

$$d(\hat{w}_k, w_k) \leq \inf_{z_k \in \mathrm{Eff}(\lambda_k)} d(\hat{w}_k, z_k) + \frac{1}{k} \leq h_k(\bar{\epsilon}_k) + \frac{1}{k}.$$

Therefore, we have the following estimations

$$0 \leq d(w_k, w_0) \leq d(w_k, \hat{w}_k) + d(\hat{w}_k, w_0) \leq h_k(\bar{\epsilon}_k) + \frac{1}{k} + d(\hat{w}_k, w_0).$$

Then, by (23) and the fact $\hat{w}_k \to w_0$, we have

$$0 \leq \limsup d(w_k, w_0) \leq \limsup \left[h_k(\bar{\epsilon}_k) + \frac{1}{k} + d(\hat{w}_k, w_0) \right] = 0.$$

Consequently, the sequence $\{w_k\}$ converges to w_0. The proof is complete. □

Remark 4.1 In [3], the authors used the following assumptions:

($\mathcal{A}1$) for any $\zeta > 0$, ψ is continuous on $[t_0, t_1] \times \zeta \overline{\mathbb{B}}_{\mathbb{R}^n} \times \Xi \times \Omega$;

($\mathcal{A}2$) for any $\zeta > 0$, there exists $\ell_\zeta \in \mathcal{L}^1([t_0, t_1], \mathbb{R})$ such that

$$|\psi(t, y_1, v, \mu) - \psi(t, y_2, v, \mu)| \leq \ell_\zeta(t) |y_1 - y_2|$$
$$\text{a.e. } t \in [t_0, t_1], \forall (y_1, v, \mu), (y_2, v, \mu) \in \zeta \overline{\mathbb{B}}_{\mathbb{R}^n} \times \Xi \times \Omega;$$

$(\mathcal{A}3)(\kappa)$ there exist a constant $\kappa \in \mathbb{N}$ and a function $\ell_\kappa \in \mathcal{L}^1([t_0, t_1], \mathbb{R})$ such that

$$|y|^{2\kappa}|\langle y, \psi(t, y, v, \mu)\rangle| \le \ell_\kappa(t)\left(1 + |y|^{2\kappa+2}\right)$$
$$\text{a.e. } t \in [t_0, t_1], \forall (y, v, \mu) \in \mathbb{R}^n \times \Xi \times \Omega.$$

Then, they established the uniform boundedness of the solution set of the Eq. (1) for all measurable function u, that is, there exists a constant ρ with

$$\rho^{2\kappa+2} := \left(|x(t_0)|^{2\kappa+2} + 2(\kappa+1)\|\ell_\kappa\|_1\right) e^{2(\kappa+1)\|\ell_\kappa\|_1}$$

such that for each ψ satisfying the conditions $(\mathcal{A}1) - (\mathcal{A}3)(\kappa)$ and for each measurable function u, the Eq. (1) has a unique solution $x(t) \in \rho\overline{\mathbb{B}}_{\mathbb{R}^n}$ for all $t \in [t_0, t_1]$. The Assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ are very common in optimal control problems.

Now, we discuss about the Assumption $(\mathcal{A}3)(\kappa)$: Frankowska and Rampazzo [21] used the following sublinear growth condition

$(\mathcal{A}3_1)$ there exists a function $\ell_1 \in \mathcal{L}^1([t_0, t_1], \mathbb{R})$, such that

$$|\psi(t, y, v, \mu)| \le \ell_1(t)(1 + |y|) \quad \text{a.e. } t \in [t_0, t_1], \forall (y, v, \mu) \in \mathbb{R}^n \times \Xi \times \Omega$$

to obtain the relative compactness of the set of solutions of a the differential inclusion, a generalization of the Eq. (1). Also, Tammer [38], and Ekeland and Temam [20] employed a bounded condition via the inner product, that is

$(\mathcal{A}3_2)$ there exists a constant ℓ_2 such that

$$|\langle y, \psi(t, y, v, \mu)\rangle| \le \ell_2(1 + |y|^2) \quad \forall (t, y, v, \mu) \in [t_0, t_1] \times \mathbb{R}^n \times \Xi \times \Omega,$$

to study the relative compactness of the set of trajectories to (1). Noting that when $\kappa = 0$, the condition $(\mathcal{A}3)(0)$ is a unified form of $(\mathcal{A}3_1)$ and $(\mathcal{A}3_2)$, and furthermore if a map ψ satisfies the Assumption $(\mathcal{A}3)(\hat{\kappa})$, then it satisfies the Assumption $(\mathcal{A}3)(\kappa)$ for all $\kappa \ge \hat{\kappa}$.

In this special case, if $\Theta \equiv \rho\overline{\mathbb{B}}_{\mathbb{R}^n}$, then the condition (ii) of the admissible control set $\mathbb{U}$ is immediately satisfied for all $u \in \mathcal{L}^p([t_0, t_1], \mathbb{R}^l)$. Moreover, if Ξ and Ω are bounded, then the set $[t_0, t_1] \times \Theta \times \Xi \times \Omega$ is compact. Then, if the maps φ and ψ are continuous on $[t_0, t_1] \times \Theta \times \Xi \times \Omega$, then they are uniformly continuous on this set, and therefore, the Hypotheses $(\mathcal{H}1)$ and $(\mathcal{H}4)$ can be restated as follows.

$(\mathcal{H}1')$ For a.e. $t \in [t_0, t_1]$, $\varphi(t, \cdot, \cdot, \cdot)$ is continuous on $\Theta \times \Xi \times \Omega$ and for each fixed $(y, v, \mu) \in \Theta \times \Xi \times \Omega$, $\varphi(\cdot, y, v, \mu)$ is measurable on $[t_0, t_1]$.

$(\mathcal{H}4')$ $\mathcal{U}$ is equimeasurable on $[t_0, t_1]$, and Ξ and Ω are bounded.

For the trajectory set Θ of the Eq. (1), which is given as in Remark 4.1, by using Theorems 4.1, 4.2 and 4.3, we have the following results on the semicontinuity of the efficient solution map.

Corollary 4.1 *Assume that the Assumptions* $(\mathcal{A}1)-(\mathcal{A}3)(\kappa)$, $(\mathcal{H}1')$ *and* $(\mathcal{H}4')$ *are fulfilled. Then, the following statements hold true.*

(a) If the Hypothesis $(\mathcal{H}5)$ *is satisfied, then the efficient solution map* Eff *is upper semicontinuous on* Λ.
(b) If the Hypothesises $(\mathcal{H}5)$ *and* $(\mathcal{H}6)$ *are satisfied, then the efficient solution map* Eff *is continuous on* Λ.
(c) If the Hypothesis $(\mathcal{H}7)$ *is satisfied on* Λ, *then the efficient solution map* Eff *is lower semicontinuous on* Λ.

5 Applications

5.1 Glucose Model

This model is used by Eisen [19], also in [30, Lab 10], in an effort to enhance the regulation of blood glucose levels in individuals with diabetes.

Denote the concentration of blood glucose by $x^{(1)}$, the net hormonal concentration by $x^{(2)}$ and the desired constant glucose level by β_0. Consider parameters $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$ and $\lambda^{(4)}$ as follows:

- $\lambda^{(1)}$: the rate of removal of glucose above the initial (fasting) level due to its own excess above the initial level,
- $\lambda^{(2)}$: the rate of removal of glucose above the initial level due to blood-hormone concentrations above the initial level,
- $\lambda^{(3)}$: the rate of removal of hormone above the initial (fasting) level due to its own excess above the initial level,
- $\lambda^{(4)}$: represents various factors such as physical activity (e.g., light exercise, moderate exercise, level of fitness/training), eating habits (e.g., carbohydrate quantity, carbohydrate type, caffeine intake, meal timing), biological factors (e.g., insufficient sleep, stress, illness, allergies).

The target of this model is to find the insulin injection level, u, which minimizes the difference between $x^{(1)}$ and β_0, while considering the cost of the treatment, i.e., find a control $u \in \mathcal{L}^p([0,t_1],\mathbb{R})$ and a state $(x^{(1)},x^{(2)})$ to minimize the functions $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ with

$$\mathcal{I}^{(1)}(x^{(1)},x^{(2)},u,\lambda)=\int_0^{t_1}\Big[f^{(1)}\big(\lambda(t)\big)\big(x^{(1)}(t)-\beta_0\big)^2+g^{(1)}\big(\lambda(t)\big)\Big]\,dt$$

$$\text{and } \mathcal{I}^{(2)}(x^{(1)},x^{(2)},u,\lambda)=\int_0^{t_1}\Big[f^{(2)}\big(\lambda(t)\big)\big(u(t)\big)^2+g^{(2)}\big(\lambda(t)\big)\Big]\,dt,$$

subject to

$$\begin{cases} \dot{x}^{(1)}(t)=-\lambda^{(1)}(t)x^{(1)}(t)-\lambda^{(2)}(t)x^{(2)}(t)-\phi^{(2)}\big(\lambda^{(4)}(t)\big),\ x^{(1)}(0)=x_{01},\\ \dot{x}^{(2)}(t)=-\lambda^{(3)}(t)x^{(2)}(t)+\phi^{(1)}\big(\lambda^{(4)}(t)\big)u(t),\ x^{(2)}(0)=0,\\ u(t)\in[-c,c] \quad t\in[0,t_1]. \end{cases}$$

Here, $\lambda := \left(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}\right)$ is an element of parameter space $\mathcal{L}^\infty([0, t_1], \mathbb{R}^4)$ satisfying $\lambda(t) \in \Omega := [0, b_1] \times [0, b_2] \times [0, b_3] \times [a_4, b_4]$ for all $t \in [0, t_1]$, and the functions $f^{(i)} : \Omega \to \mathbb{R}_{++}$, $g^{(i)} : \Omega \to \mathbb{R}$ and $\phi^{(i)} : [a_4, b_4] \to \mathbb{R}_+$ with $i \in \{1, 2\}$ are continuous, where $\mathbb{R}_{++} := \{a \in \mathbb{R} : a > 0\}$. Setting

$$\begin{aligned}
&\Lambda := \{\lambda \in \mathcal{L}^\infty([0, t_1], \mathbb{R}^4) : \lambda(t) \in \Omega,\ \forall t \in [0, t_1]\},\ \Xi := [-c, c],\\
&x := (x^{(1)}, x^{(2)})^T, x^0 := (x_{01}, 0)^T,
\end{aligned}$$

$$\begin{aligned}
\psi(t, x(t), u(t), \lambda(t)) := \begin{pmatrix} -\lambda^{(1)}(t) & -\lambda^{(2)}(t) \\ 0 & -\lambda^{(3)}(t) \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \phi^{(1)}\big(\lambda^{(4)}(t)\big) \end{pmatrix} u(t)\\
+ \begin{pmatrix} -\phi^{(2)}\big(\lambda^{(4)}(t)\big) \\ 0 \end{pmatrix}.
\end{aligned}$$

It is easy to see that the function ψ satisfies the Assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ in Remark 4.1. Moreover, due to the continuity of $\phi^{(i)}$ on $[a_4, b_4]$, there exist $\alpha_i > 0$ such that $\phi^{(i)}(\mu) \le \alpha_i$ for all $\mu \in [a_4, b_4]$ and $i \in \{1, 2\}$. Then, for all $(t, y, v, \mu) \in [0, t_1] \times \mathbb{R}^2 \times \Xi \times \Omega$, we have

$$\begin{aligned}
|\langle y, \psi(t, y, v, \mu)\rangle| \le |y||\psi(t, y, v, \mu)| &\le |y|\left(\max\{b_1, b_2, b_3\}|y| + \alpha_1|v| + \alpha_2\right)\\
&\le \max\{b_1, b_2, b_3, \alpha_1 c, \alpha_2\}\left(|y|^2 + 2|y|\right)\\
&\le 2\max\{b_1, b_2, b_3, \alpha_1 c, \alpha_2\}\left(|y|^2 + |y|\right)\\
&\le 4\max\{b_1, b_2, b_3, \alpha_1 c, \alpha_2\}\left(|y|^2 + 1\right).
\end{aligned}$$

Therefore, ψ satisfies the Assumption $(\mathcal{A}3)(0)$ with $\ell_0 = 4\max\{b_1, b_2, b_3, \alpha_1 c, \alpha_2\}$. Hence, this glucose model is a special case of the problem $(\mathcal{P})$ with $n = 2, m = 2, l = 1, r = 1$,

$$\begin{aligned}
\varphi(t, x(t), u(t), \lambda(t)) :=\Big(& f^{(1)}\big(\lambda(t)\big)\big(x^{(1)}(t) - \beta_0\big)^2 + g^{(1)}\big(\lambda(t)\big),\\
& f^{(2)}\big(\lambda(t)\big)\big(u(t)\big)^2 + g^{(2)}\big(\lambda(t)\big)\Big),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(w, \lambda) :=\Big(& \int_0^{t_1} \Big[f^{(1)}\big(\lambda(t)\big)\big(x^{(1)}(t) - \beta_0\big)^2 + g^{(1)}\big(\lambda(t)\big)\Big]dt,\\
& \int_0^{t_1} \Big[f^{(2)}\big(\lambda(t)\big)\big(u(t)\big)^2 + g^{(2)}\big(\lambda(t)\big)\Big]dt\Big),
\end{aligned}$$

and $K(\lambda) := \left\{w = (x, u) \in \mathcal{X} \times \mathcal{U} : x(t) = x^0 + \int_0^t \psi(s, x(s), u(s), \lambda(s))ds\right\}.$

Then, the above problem can be cast as the following form:

$$\min \mathcal{I}(w, \lambda) \text{ subject to } w \in K(\lambda).$$

The following result is obtained from Corollary 4.1.

Corollary 5.1 *If $\mathcal{U}$ is convex and equimeasurable on* $[0, t_1]$*, then the efficient solution map of the glucose model,* $\mathrm{Eff}(GM)$*, is continuous on* Λ.

Proof It is clear that the Hypotheses ($\mathcal{H}1'$) and ($\mathcal{H}4'$) of Corollary 4.1 are satisfied. So, in order to apply Corollary 4.1, we now check the Hypotheses ($\mathcal{H}5$) and ($\mathcal{H}6$) of Corollary 4.1. For each $(t, y, v, \mu) \in [0, t_1] \times \rho\overline{\mathbb{B}}_{\mathbb{R}^2} \times \Xi \times \Omega$ and $y = (y^{(1)}, y^{(2)})$, we set

$$\begin{aligned}
&\varphi^{(1)}(y, \mu) := f^{(1)}(\mu)\big(y^{(1)} - \beta_0\big)^2 + g^{(1)}(\mu), \text{ and} \\
&\varphi^{(2)}(v, \mu) := f^{(2)}(\mu)v^2 + g^{(2)}(\mu),
\end{aligned}$$

where $\rho = \big(|x_{01}|^2 + 2\ell_0 t_1\big)\, e^{2\ell_0 t_1}$. Then, we have $\varphi(t, y, v, \mu) = \big(\varphi^{(1)}(y, \mu), \varphi^{(2)}(v, \mu)\big)$. Obviously, the functions $\varphi^{(1)}$ and $\varphi^{(2)}$ are strictly convex in the first component on convex subsets $\rho\overline{\mathbb{B}}_{\mathbb{R}^2}$ and Ξ, respectively.

We now show that $K(\lambda)$ is convex for any $\lambda \in \Lambda$. Let $w_1 = (x_1, u_1)$, $w_2 = (x_2, u_2) \in K(\lambda)$ and $\tau \in [0, 1]$ be arbitrary, we have

$$\begin{aligned}
&\frac{d}{dt}(\tau x_1(t) + (1-\tau)x_2(t)) = \tau \dot{x}_1(t) + (1-\tau)\dot{x}_2(t) \\
&= \tau \begin{pmatrix} -\lambda^{(1)}(t) & -\lambda^{(2)}(t) \\ 0 & -\lambda^{(3)}(t) \end{pmatrix} x_1(t) + \tau \begin{pmatrix} 0 \\ \phi^{(1)}\big(\lambda^{(4)}(t)\big) \end{pmatrix} u_1(t) + \tau \begin{pmatrix} -\phi^{(2)}\big(\lambda^{(4)}(t)\big) \\ 0 \end{pmatrix} \\
&\quad + (1-\tau) \begin{pmatrix} -\lambda^{(1)}(t) & -\lambda^{(2)}(t) \\ 0 & -\lambda^{(3)}(t) \end{pmatrix} x_2(t) + (1-\tau) \begin{pmatrix} 0 \\ \phi^{(1)}\big(\lambda^{(4)}(t)\big) \end{pmatrix} u_2(t) \\
&\quad + (1-\tau) \begin{pmatrix} -\phi^{(2)}\big(\lambda^{(4)}(t)\big) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\lambda^{(1)}(t) & -\lambda^{(2)}(t) \\ 0 & -\lambda^{(3)}(t) \end{pmatrix} \big(\tau x_1(t) + (1-\tau)x_2(t)\big) \\
&\quad + \begin{pmatrix} 0 \\ \phi^{(1)}\big(\lambda^{(4)}(t)\big) \end{pmatrix} \big(\tau u_1(t) + (1-\tau)u_2(t)\big) + \begin{pmatrix} -\phi^{(2)}\big(\lambda^{(4)}(t)\big) \\ 0 \end{pmatrix}.
\end{aligned} \tag{24}$$

Due to the convexity of $\mathcal{U}$, one gets $\tau u_1 + (1-\tau)u_2 \in \mathcal{U}$. This together with (24) implies that $\tau w_1 + (1-\tau)w_2 \in K(\lambda)$. So, $K(\lambda)$ is convex.

Now, let $\vartheta \in \mathbb{R}^2$, and $w_1 = (x_1, u_1)$, $w_2 = (x_2, u_2) \in K(\lambda)$ such that

$$\int_0^{t_1} \varphi\big(t, x_1(t), u_1(t), \lambda(t)\big)dt \leq_{\mathbb{R}^2_+} \vartheta \text{ and } \int_0^{t_1} \varphi\big(t, x_2(t), u_2(t), \lambda(t)\big)dt \leq_{\mathbb{R}^2_+} \vartheta. \tag{25}$$

Then, for all $\tau \in [0, 1]$, we have $\tau w_1 + (1-\tau)w_2 \in K(\lambda)$ and

$$\begin{aligned}
&\varphi\big(t, \tau w_1(t) + (1-\tau)w_2(t), \lambda(t)\big) \\
&\quad = \Big(\varphi^{(1)}\big(\tau x_1(t) + (1-\tau)x_2(t), \lambda(t)\big), \varphi^{(2)}\big(\tau u_1(t) + (1-\tau)u_2(t), \lambda(t)\big)\Big) \\
&\quad <_{\mathbb{R}^2_+} \Big(\tau\varphi^{(1)}\big(x_1(t), \lambda(t)\big) + (1-\tau)\varphi^{(1)}\big(x_2(t), \lambda(t)\big), \\
&\qquad\qquad \tau\varphi^{(2)}\big(u_1(t), \lambda(t)\big) + (1-\tau)\varphi^{(2)}\big(u_2(t), \lambda(t)\big)\Big) \\
&\quad = \tau\Big(\varphi^{(1)}\big(x_1(t), \lambda(t)\big), \varphi^{(2)}\big(u_1(t), \lambda(t)\big)\Big) \\
&\qquad + (1-\tau)\Big(\varphi^{(1)}\big(x_2(t), \lambda(t)\big), \varphi^{(2)}\big(u_2(t), \lambda(t)\big)\Big) \\
&\quad = \tau\varphi\big(t, x_1(t), u_1(t), \lambda(t)\big) + (1-\tau)\varphi\big(t, x_2(t), u_2(t), \lambda(t)\big).
\end{aligned} \tag{26}$$

We now consider the continuous map $\xi : [0, 1] \to K(\lambda)$ defined by $\xi(\tau) = \tau w_1 + (1-\tau)w_2$. Then, $\xi(0) = w_2$, $\xi(1) = w_1$ and for all $\tau \in [0, 1]$, it follows from (26) that

$$\begin{aligned}
\int_0^{t_1} \varphi(t, \xi(\tau)(t), \lambda(t))dt \leq_{\mathbb{R}^2_+} \tau &\int_0^{t_1} \varphi(t, x_1(t), u_1(t), \lambda(t))dt \\
&+ (1-\tau)\int_0^{t_1} \varphi(t, x_2(t), u_2(t), \lambda(t))dt.
\end{aligned}$$

Combining this with (25), we have

$$\int_0^{t_1} \varphi(t, \xi(\tau)(t), \lambda(t))dt \leq_{\mathbb{R}^2_+} \tau\vartheta + (1-\tau)\vartheta = \vartheta.$$

So, for each $t \in [t_0, t_1]$, $\varphi(t, \cdot, \cdot, \lambda(t))$ is $\mathbb{R}^m_+$-quasi-arcwise connected integrand with respect to $K(\lambda)$. Therefore, the Hypothesis $(\mathcal{H}6)$ of Corollary 4.1 is satisfied.

Furthermore, let $\bar{w} = (\bar{x}, \bar{u})$, $\bar{\bar{w}} = (\bar{\bar{x}}, \bar{\bar{u}}) \in K(\lambda)$ be arbitrary with $\bar{w} \neq \bar{\bar{w}}$. Then, we have also $\frac{1}{2}\bar{w}(t) + \frac{1}{2}\bar{\bar{w}}(t) \in K(\lambda)$ and

$$\varphi\Big(t, \frac{1}{2}\bar{w}(t) + \frac{1}{2}\bar{\bar{w}}(t), \lambda(t)\Big) <_{\mathbb{R}^2_+} \frac{1}{2}\varphi(t, \bar{x}(t), \bar{u}(t), \lambda(t)) + \frac{1}{2}\varphi(t, \bar{\bar{x}}(t), \bar{\bar{u}}(t), \lambda(t)).$$

Equivalently,

$$\begin{aligned}
\varphi\big(t, \frac{1}{2}\bar{w}(t) + \frac{1}{2}\bar{\bar{w}}(t), \lambda(t)\big) &- \frac{1}{2}\varphi(t, \bar{x}(t), \bar{u}(t), \lambda(t)) \\
&- \frac{1}{2}\varphi(t, \bar{\bar{x}}(t), \bar{\bar{u}}(t), \lambda(t)) \in -\operatorname{int}\mathbb{R}^2_+.
\end{aligned}$$

It follows that

$$\int_0^{t_1} \varphi\Big(t, \frac{1}{2}\bar{w}(t) + \frac{1}{2}\bar{\bar{w}}(t), \lambda(t)\Big)dt - \frac{1}{2}\int_0^{t_1} \varphi(t, \bar{x}(t), \bar{u}(t), \lambda(t))dt \\ - \frac{1}{2}\int_0^{t_1} \varphi(t, \bar{\bar{x}}(t), \bar{\bar{u}}(t), \lambda(t))dt \in -\operatorname{int}\mathbb{R}^2_+.$$

So, for each $t \in [0, t_1]$, $\varphi(t, \cdot, \cdot, \lambda(t))$ is strictly natural $\mathbb{R}^m_+$-quasi-convex-like integrand with respect to $K(\lambda)$, and hence the Hypothesis $(\mathcal{H}5)$ of Corollary 4.1 is satisfied. According to Corollary 4.1, the efficient solution map $\mathrm{Eff}(GM)$ is continuous on Λ. □

We will finalize this work with considering an important special case of our multi-objective optimal control problem discussed in the second section.

5.2 Epidemic Model

This model uses optimal control techniques to establish a vaccination schedule for managing an epidemic disease, specifically focusing on a micro-parasitic infectious disease (see [30, Lab 7]).

Consider $x^{(1)}(t)$, $x^{(2)}(t)$ and $x^{(3)}(t)$ as representing the number of susceptible, infectious, and recovered individuals at time t. The model accommodates an incubation period for the disease within the host, during which an infected individual remains in a latent state before transitioning to infectious, thereby forming an exposed category. Let $x^{(4)}(t)$ denote the number of latent or exposed individuals at time t and $x^{(5)}(t)$ denote the overall population size, such that $x^{(5)}(t) = x^{(1)}(t) + x^{(2)}(t) + x^{(3)}(t) + x^{(4)}(t)$ and $x^{(5)}(t) \le \rho$ for all $t \in [0, t_1]$ and for some $\rho \in \mathbb{R}_+$. Define $u(t)$ as the control variable representing the percentage of susceptible individuals being vaccinated per unit of time. Consider parameters $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, $\lambda^{(4)}$, $\lambda^{(5)}$, $\lambda^{(6)}$ and $\lambda^{(7)}$ as follows:

- $\lambda^{(1)}$: the death rate among infectious individuals due to the disease,
- $\lambda^{(2)}$: the natural exponential birth rate of the population,
- $\lambda^{(3)}$: the incidence of the disease,
- $\lambda^{(4)}$: the natural exponential death rate,
- $\lambda^{(5)}$: the rate at which exposed individuals transition to the infectious state,
- $\lambda^{(6)}$: the rate at which infectious individuals recover,
- $\lambda^{(7)}$: represents various factors affect an epidemic in different ways such as weather information, incoming and outgoing population of the area, agent density in the area, etc.

The target of this model is to find the percentage of susceptible individuals being vaccinated, which minimizes both the number of infectious individuals and the overall vaccine cost over a fixed time period, i.e., find a control $u \in \mathcal{L}^p([0, t_1]; \mathbb{R})$ and a state $\big(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}\big)$ to minimize the functions $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ with

$$\mathcal{I}^{(1)} = \int_0^{t_1} \Big[f^{(1)}\big(\lambda(t)\big)x^{(2)}(t) + g^{(1)}\big(\lambda(t)\big)\Big]dt$$

$$\text{and } \mathcal{I}^{(2)} = \int_0^{t_1} \left[f^{(2)}\big(\lambda(t)\big)\big(u(t)\big)^2 + g^{(2)}\big(\lambda(t)\big) \right] dt,$$

subject to

$$\begin{cases} \dot{x}^{(1)}(t) = \lambda^{(2)}(t)x^{(5)}(t) - \lambda^{(4)}(t)x^{(1)}(t) - \lambda^{(3)}(t)x^{(1)}(t)x^{(2)}(t) - u(t)x^{(1)}(t) \\ \qquad\qquad + \phi^{(1)}\big(\lambda^{(7)}(t)\big), x^{(1)}(0) = x_{01} \geq 0, \\ \dot{x}^{(2)}(t) = \lambda^{(5)}(t)x^{(4)}(t) - \Big(\lambda^{(6)}(t) + \lambda^{(1)}(t) + \lambda^{(4)}(t)\Big) x^{(2)}(t) + \phi^{(2)}\big(\lambda^{(7)}(t)\big), \\ \qquad\qquad x^{(2)}(0) = x_{02} \geq 0, \\ \dot{x}^{(3)}(t) = \lambda^{(6)}(t)x^{(2)}(t) - \lambda^{(4)}(t)x^{(3)}(t) + u(t)x^{(1)}(t) + \phi^{(3)}\big(\lambda^{(7)}(t)\big), \\ \qquad\qquad x^{(3)}(0) = x_{03} \geq 0, \\ \dot{x}^{(4)}(t) = \lambda^{(3)}(t)x^{(1)}(t)x^{(2)}(t) - \Big(\lambda^{(4)}(t) + \lambda^{(5)}(t)\Big) x^{(4)}(t) + \phi^{(4)}\big(\lambda^{(7)}(t)\big), \\ \qquad\qquad x^{(4)}(0) = x_{04} \geq 0, \\ \dot{x}^{(5)}(t) = \Big(\lambda^{(2)}(t) - \lambda^{(4)}(t)\Big) x^{(5)}(t) - \lambda^{(1)}(t)x^{(2)}(t) + \phi^{(5)}\big(\lambda(t)\big), x^{(5)}(0) = x_{05}, \\ 0 \leq u(t) \leq 0.9. \end{cases}$$

Herein, $\lambda = \big(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}, \lambda^{(5)}, \lambda^{(6)}, \lambda^{(7)}\big)$ is an element of the parameter space $\mathcal{L}^\infty([0, t_1], \mathbb{R}^7)$ satisfying $\lambda(t) \in \Omega := [0, b_1] \times [0, b_2] \times [0, b_3] \times [0, b_4] \times [0, b_5] \times [0, b_6] \times [a_7, b_7]$ for all $t \in [0, t_1]$, and the functions $f^{(i)}, g^{(i)}, \phi^{(5)} : \Omega \to \mathbb{R}$ with $i \in \{1, 2\}$ and the functions $\phi^{(j)} : [a_7, b_7] \to \mathbb{R}$ with $j \in \{1, 2, 3, 4\}$ are continuous.

Then, this model is also a special case of $(\mathcal{P})$ with $n = 5$, $m = 2$, and let

$$\begin{aligned} &\mathcal{X} := \mathcal{C}([0, t_1], \mathbb{R}^5),\ \Xi := [0, 0.9],\ \Theta := \rho\overline{\mathbb{B}}_{\mathbb{R}^5}, \\ &\Lambda := \left\{\lambda \in \mathcal{L}^\infty([0, t_1], \mathbb{R}^7) : \lambda(t) \in \Omega,\ \forall t \in [0, t_1]\right\}, \\ &x := \left(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}\right)^T,\ x^0 := (x_{01}, x_{02}, x_{03}, x_{04}, x_{05})^T, \\ &\psi(t, x(t), u(t), \lambda(t)) \\ &:= \begin{pmatrix} \lambda^{(2)}(t)x^{(5)}(t) - \lambda^{(4)}(t)x^{(1)}(t) - \lambda^{(3)}(t)x^{(1)}(t)x^{(2)}(t) - u(t)x^{(1)}(t) + \phi^{(1)}\big(\lambda^{(7)}(t)\big) \\ \lambda^{(5)}(t)x^{(4)}(t) - \Big(\lambda^{(6)}(t) + \lambda^{(1)}(t) + \lambda^{(4)}(t)\Big) x^{(2)}(t) + \phi^{(2)}\big(\lambda^{(7)}(t)\big) \\ \lambda^{(6)}(t)x^{(2)}(t) - \lambda^{(4)}(t)x^{(3)}(t) + u(t)x^{(1)}(t) + \phi^{(3)}\big(\lambda^{(7)}(t)\big) \\ \lambda^{(3)}(t)x^{(1)}(t)x^{(2)}(t) - \Big(\lambda^{(4)}(t) + \lambda^{(5)}(t)\Big) x^{(4)}(t) + \phi^{(4)}\big(\lambda^{(7)}(t)\big) \\ \Big(\lambda^{(2)}(t) - \lambda^{(4)}(t)\Big) x^{(5)}(t) - \lambda^{(1)}(t)x^{(2)}(t) + \phi^{(5)}\big(\lambda(t)\big) \end{pmatrix}, \\ &\varphi(t, x(t), u(t), \lambda(t)) \\ &\quad := \left(f^{(1)}\big(\lambda(t)\big)x^{(2)}(t) + g^{(1)}\big(\lambda(t)\big),\ f^{(2)}\big(\lambda(t)\big)\big(u(t)\big)^2 + g^{(2)}\big(\lambda(t)\big)\right), \end{aligned}$$

$$\mathcal{I}(x, u, \lambda) := \int_0^{t_1} \varphi(t, x(t), u(t), \lambda(t)) dt,$$

and

$$K(\lambda) := \left\{ w = (x, u) \in \mathcal{X} \times \mathcal{U} : x(t) = x^0 + \int_0^t \psi(s, x(s), u(s), \lambda(s)) ds \right\}.$$

Then, the above problem can be cast as the following form:

$$\min \mathcal{I}(w, \lambda) \text{ subject to } w \in K(\lambda).$$

In this model, the feasible solution set $K(\lambda)$ is nonconvex, even though it is arcwise connected when the admissible control set $\mathcal{U}$ is arcwise connected, as stated in Theorem 3.1. However, the multiobjective function $\mathcal{I}(x, u, \lambda)$ is linear in the first component, and hence the strictly quasi-arcwise connected integrand condition at the original point as required in the Hypothesis ($\mathcal{H}6$) is not true in general. Therefore, we will consider the lower semicontinuity property of the efficient solution map of this model by using Theorem 4.3 instead of Theorem 4.2.

Corollary 5.2 *Assume that $\mathcal{U}$ is equimeasurable on $[0, t_1]$. Then, the efficient solution map of the epidemic model,* Eff(EM)*, is lower semicontinuous on Λ if the Hypothesis ($\mathcal{H}7$) is satisfied on Λ.*

Proof Because the maps φ and ψ are continuous on the compact set $[t_0, t_1] \times \Theta \times \Xi \times \Omega$, they are uniformly continuous on this set. So, the Hypotheses ($\mathcal{H}1$), ($\mathcal{H}2$) and ($\mathcal{H}4$) are satisfied as $\mathcal{U}$ is equimeasurable on $[0, t_1]$. Now, we check the Hypothesis ($\mathcal{H}3$). For all $(t, y, v, \mu), (t, \hat{y}, v, \mu) \in [t_0, t_1] \times \Theta \times \Xi \times \Omega$, where

$$\begin{aligned} y &= \left(y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}\right), \\ \hat{y} &= \left(\hat{y}^{(1)}, \hat{y}^{(2)}, \hat{y}^{(3)}, \hat{y}^{(4)}, \hat{y}^{(5)}\right), \\ \text{and } \mu &= \left(\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \mu^{(4)}, \mu^{(5)}, \mu^{(6)}, \mu^{(7)}\right), \end{aligned}$$

by setting

$$\begin{aligned} \beta_1 &= \mu^{(2)}\left(y^{(5)} - \hat{y}^{(5)}\right) - \mu^{(4)}\left(y^{(1)} - \hat{y}^{(1)}\right) - \mu^{(3)}\left(y^{(1)}y^{(2)} - \hat{y}^{(1)}\hat{y}^{(2)}\right) \\ &\qquad - v\left(y^{(1)} - \hat{y}^{(1)}\right), \\ \beta_2 &= \mu^{(5)}\left(y^{(4)} - \hat{y}^{(4)}\right) - \left(\mu^{(6)} + \mu^{(1)} + \mu^{(4)}\right)\left(y^{(2)} - \hat{y}^{(2)}\right), \\ \beta_3 &= \mu^{(6)}\left(y^{(2)} - \hat{y}^{(2)}\right) - \mu^{(4)}\left(y^{(3)} - \hat{y}^{(3)}\right) + v\left(y^{(1)} - \hat{y}^{(1)}\right), \\ \beta_4 &= \mu^{(3)}\left(y^{(1)}y^{(2)} - \hat{y}^{(1)}\hat{y}^{(2)}\right) - \left(\mu^{(4)} + \mu^{(5)}\right)\left(y^{(4)} - \hat{y}^{(4)}\right), \\ \beta_5 &= \left(\mu^{(2)} - \mu^{(4)}\right)\left(y^{(5)} - \hat{y}^{(5)}\right) - \mu^{(1)}\left(y^{(2)} - \hat{y}^{(2)}\right), \end{aligned}$$

we have the following estimations

$$\begin{aligned}
|\beta_1| &= \left|\mu^{(2)}\left(y^{(5)}-\hat{y}^{(5)}\right)\right| + \left|\mu^{(4)}\left(y^{(1)}-\hat{y}^{(1)}\right)\right| + \left|\mu^{(3)}\left(y^{(1)}y^{(2)}-\hat{y}^{(1)}\hat{y}^{(2)}\right)\right| \\
&\quad + \left|v\left(y^{(1)}-\hat{y}^{(1)}\right)\right| \\
&\le b_2\left|y^{(5)}-\hat{y}^{(5)}\right| + b_4\left|y^{(1)}-\hat{y}^{(1)}\right| \\
&\quad + b_3\left|y^{(1)}\left(y^{(2)}-\hat{y}^{(2)}\right)+\hat{y}^{(2)}\left(y^{(1)}-\hat{y}^{(1)}\right)\right| \\
&\quad + 0.9\left|y^{(1)}-\hat{y}^{(1)}\right| \\
&\le (b_2+b_4+2\rho b_3+0.9)\left|y-\hat{y}\right|, \\
|\beta_2| &= \left|\mu^{(5)}\left(y^{(4)}-\hat{y}^{(4)}\right) - \left(\mu^{(6)}+\mu^{(1)}+\mu^{(4)}\right)\left(y^{(2)}-\hat{y}^{(2)}\right)\right| \\
&\le (b_5+b_6+b_1+b_4)\left|y-\hat{y}\right|, \\
|\beta_3| &= \left|\mu^{(6)}\left(y^{(2)}-\hat{y}^{(2)}\right) - \mu^{(4)}\left(y^{(3)}-\hat{y}^{(3)}\right) + v\left(y^{(1)}-\hat{y}^{(1)}\right)\right| \\
&\le (b_6+b_4+0.9)\left|y-\hat{y}\right|, \\
|\beta_4| &= \left|\mu^{(3)}\left(y^{(1)}y^{(2)}-\hat{y}^{(1)}\hat{y}^{(2)}\right) - \left(\mu^{(4)}+\mu^{(5)}\right)\left(y^{(4)}-\hat{y}^{(4)}\right)\right| \\
&\le (2\rho b_3+b_4+b_5)\left|y-\hat{y}\right|, \\
|\beta_5| &= \left|\left(\mu^{(2)}-\mu^{(4)}\right)\left(y^{(5)}-\hat{y}^{(5)}\right) - \mu^{(1)}\left(y^{(2)}-\hat{y}^{(2)}\right)\right| \\
&\le (b_2+b_4+b_1)\left|y-\hat{y}\right|.
\end{aligned}$$

Then, we obtain

$$\left|\psi(t,y,v,\mu)-\psi(t,\hat{y},v,\mu)\right| = \max_i |\beta_i| \le \ell|y-\hat{y}|,$$

where $\ell := \max\{b_2+b_4+2\rho b_3+0.9, b_5+b_6+b_1+b_4, b_6+b_4+0.9, b_3+b_4+b_5, b_2+b_4+b_1\}$, and hence the Hypothesis $(\mathcal{H}3)$ is satisfied. Applying Theorem 4.3, the efficient solution map of the epidemic model is lower semicontinuous on Λ. $\square$

6 Concluding Remarks

In this paper, by using the concept of equimeasurability and its properties given in Chang [12] of the admissible control set, we established the compactness and arcwise connectedness of the feasible solution set. Combining these results with the concept of convex integrand introduced by Rockafellar [36], we proposed concepts of the quasi-arcwise connected integrand and employed them to study the semicontinuity of efficient solution map. In the case that the multiobjective function did not satisfy these properties, picking up ideas of Zhao [42] and Kien [24] about the estimation conditions of approximate solutions, we introduced a key hypothesis concerning the epi-convergence condition introduced by Rockafellar and Wets [37], and we used it to formulate the lower semicontinuity of efficient solution map. To the best of our knowledge, the results obtained in this paper are the pioneering ones about continuous conditions for the parametric nonlinear nonconvex multiobjective optimal control problems without assuming any condition related to the differential of multiobjective function. Therefore, the techniques and approaches of this work have genuine potential in studying solution properties for optimal control models.

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Declarations

Conflict of interest No potential Conflict of interest was reported by the authors.

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