Co-ordinated convexity according to a pair of quasi-arithmetic means on the rectangle from plane and inequalities

Dinh Thanh Duc[a], Tran Dinh Phung[b,*], Nguyen Du Vi Nhan[c,a], Vu Kim Tuan[d]

[a]*Department of Mathematics and Statistics, Quy Nhon University, Binh Dinh, Vietnam*
[b]*Department of Mathematics and Statistics, University of Finance-Marketing, Ho Chi Minh, Vietnam*
[c]*Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA*
[d]*Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA*

Abstract

We consider a class of generalized convex functions, which are defined according to a pair of quasi-arithmetic means on the rectangle from the plane and called co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, and establish various Fejér type inequalities for such a function class. Applications to inequalities involving the gamma function, the beta function, the fractional functions and special means are also included.

Keywords: Quasi-arithmetic mean, convexity, Hermite-Hadamard inequality, Fejér inequality, co-ordinated convexity, gamma function
2010 MSC: 26A51, 26A33, 26D10, 26D15, 33B15

1. Introduction

The Hermite-Hadamard inequality, name after Charles Hermite [21] and Jacques Hadamard [19] and sometimes also called Hadamard's inequality gives us a lower and an upper estimations for the integral mean value of any convex function on a closed interval, involving the midpoint and the endpoints of the domain. More precisely, if $f : [a, b] \to \mathbb{R}$ is convex, then the following chain of inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

There is an extensive literature devoted to develop applications of this inequality, as well as to discuss its extensions, by considering other measures, other kinds of convexity, or higher dimensions (see, for example, [4, 7, 9, 12, 13, 16, 17, 20, 24, 25, 29, 30, 34, 35, 45, 46, 47, 48, 49, 50, 42, 43]). Many classical results related to this inequality can be found in the

*Corresponding author
Email addresses: dinhthanhduc@qnu.edu.vn (Dinh Thanh Duc), td.phung@ufm.edu.vn (Tran Dinh Phung), ndn3em@virginia.edu, ndvynhan@gmail.com (Nguyen Du Vi Nhan), vu@westga.edu (Vu Kim Tuan)

monograph of Pečarić, Proschan and Tong [38]. Especially, in the last two decades it has received much attention. The monograph of Dragomir and Pearce [14] gives a comprehensive review of this literature.

In recent years, many mathematicians have studied the results for the inequalities for co -ordinated convex functions [14, 22, 36, 23, 42, 39]. Especially, in [10], Dragomir established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a retangle from the plane $\mathbb{R}^2$.

Theorem 1.1. *Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\ & \leq \frac{1}{4}\left[\frac{1}{b-a}\int_c^d f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ & \qquad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ & \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.2}$$

These inequalities are sharp.

Recently, Duc, Hue, Nhan and Tuan [8] was consider a class of generalized convex functions which are defined according to a pair of quasi-arithmetic means and called $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, and establish various Fejér type inequalities for such a function class.

In this paper, we investigate weighted quasi-arithmetic means on the rectangle from the plane.

Let $\phi : [a, b] \to \mathbb{R}$ a continuous and strictly monotonic function. The quasi-arithmetic mean of a and b with weight $\alpha \in [0, 1]$ is denoted by $\mathcal{M}_\phi(a, b; \alpha)$ and defined by [5]

$$\mathcal{M}_\phi(a, b; \alpha) = \phi^{-1}\left(\alpha\phi(a) + (1-\alpha)\phi(b)\right).$$

Here and subsequently, E and J denote open intervals in the real line $\mathbb{R}$, $\phi : E \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ are continuous and strictly monotonic functions. With the quasi-arithmetic means $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$ in hand, we are now in a position to generalize the notion of convexity. According to Aumann [2] (see also [33]), a function $f : E \to J$ is said to be $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex if it verifies the following analogue of Jensen's inequality:

$$f(\mathcal{M}_\phi(x, y; \alpha)) \leq \mathcal{M}_\psi(f(x), f(y); \alpha) \tag{1.3}$$

for all $x, y \in E$ and $\alpha \in [0,1]$.

Let us consider the bidimensional interval $I := [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$ with $a_1 < b_1, a_2 < b_2$ and $[a_1, b_1] \subset E, [a_2, b_2] \subset E$, E and J are open intervals in the real line $\mathbb{R}$, $\phi : E \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ are continuous and strictly monotonic functions.

A function $f : I \to J$ is co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex on I if the partial mappings

$$f_y : [a_1, b_1] \to J, \quad f_y(u) := f(u, y)$$

and

$$f_x : [a_2, b_2] \to J, \quad f_x(v) := f(x, v)$$

are $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex for all $y \in [a_1, b_1]$ and $x \in [a_2, b_2]$.

Note that every $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex mapping $f : I \to J$ is co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex but the converses not generally true.

Accordingly, the aim of the present paper is to deal with interpolating inequalities of Fejér, which not only provide a natural and intrinsic characterization of the co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions, but also agree with a wide class of known inequalities of Hermite-Hadamard and Fejér type for different kinds of convexity. At the same time, we establish some inequalities involving the gamma function, the beta function, the continuous probability density functions.

2. Fejér type inequalities for co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions

In what follows, let $f : I \to J$ be co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex on I, and let $\omega_i, \theta_i : [0,1] \to [0, \infty)$ be integrable, with $\int_0^s \omega_i(t)dt > 0; \int_s^1 \theta_i(t)dt > 0, i = 1, 2$ for all $s \in (0,1)$.

For simplicity of notation, we will write

$$\begin{aligned}
&a = (a_1, a_2), b = (b_1, b_2), t = (t_1, t_2) \in [0,1]^2, \\
&\mathcal{M}_\phi(a, b; \alpha) = (\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)), \\
&\mathcal{L}_1(t_1) = \mathcal{M}_\phi(a_1, \mathcal{M}_\phi(a_1, b_1; \alpha); t_1), \\
&\mathcal{L}_2(t_2) = \mathcal{M}_\phi(a_2, \mathcal{M}_\phi(a_2, b_2; \alpha); t_2), \\
&\mathcal{L}(t) = \mathcal{L}(t_1, t_2) = \Big(\mathcal{L}_1(t_1); \mathcal{L}_2(t_2)\Big), \\
&\mathcal{R}_1(t_1) = \mathcal{M}_\phi(b_1, \mathcal{M}_\phi(a_1, b_1; \alpha); t_1), \\
&\mathcal{R}_2(t_2) = \mathcal{M}_\phi(b_2, \mathcal{M}_\phi(a_2, b_2; \alpha); t_2), \\
&\mathcal{R}(t) = \mathcal{R}(t_1, t_2) = \Big(\mathcal{R}_1(t_1); \mathcal{R}_2(t_2)\Big).
\end{aligned}$$

We see that

$$\begin{aligned}
&\mathcal{L}(1,1) = (a_1, a_2), && \mathcal{R}(1,1) = (b_1, b_2), \\
&\mathcal{L}(1,0) = (a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)), && \mathcal{R}(1,0) = (b_1, \mathcal{M}_\phi(a_2, b_2; \alpha)), \\
&\mathcal{L}(0,1) = (\mathcal{M}_\phi(a_1, b_1; \alpha), a_2), && \mathcal{R}(0,1) = (\mathcal{M}_\phi(a_1, b_1; \alpha), b_2), \\
&\mathcal{L}(0,0) = (\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)), && \mathcal{R}(0,0) = (\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)).
\end{aligned}$$

Theorem 2.1. *Let* $\mathcal{F}, \mathcal{G} : [0,1]^2 \to \mathbb{R}$ *be defined by*

$$\mathcal{F}(t) = \psi^{-1}\Big[\alpha^2 \psi \circ f(\mathcal{L}_1(t_1), \mathcal{L}_2(t_2)) + \alpha(1-\alpha)\psi \circ f(\mathcal{L}_1(t_1), \mathcal{R}_2(t_2)) + (1-\alpha)\alpha\psi \circ f(\mathcal{R}_1(t_1), \mathcal{L}_2(t_2)) + (1-\alpha)^2 \psi \circ f(\mathcal{R}_1(t_1), \mathcal{R}_2(t_2))\Big]$$

and

$$\mathcal{G}(t) = \psi^{-1}\Big[t_1 t_2 \psi \circ \mathcal{F}(1,1) + t_1(1-t_2)\psi \circ \mathcal{F}(1,0) + (1-t_1)t_2\psi \circ \mathcal{F}(0,1) + (1-t_1)(1-t_2)\psi \circ \mathcal{F}(0,0)\Big]$$

respectively.

1. *The functions* $\mathcal{F}$ *and* $\mathcal{G}$ *are* $\mathcal{M}_\psi$*-convex, increasing on co-ordinates on* $[0,1]^2$*, and*

$$\begin{aligned}
&\mathcal{F}(0,0) = \mathcal{G}(0,0) = f(\mathcal{M}_\phi(a,b;\alpha)),\\
&\mathcal{F}(1,0) = \mathcal{G}(1,0) = \mathcal{M}_\psi[f(a_1, \mathcal{M}_\phi(a_2,b_2;\alpha)), f(b_1, \mathcal{M}_\phi(a_2,b_2;\alpha));\alpha],\\
&\mathcal{F}(0,1) = \mathcal{G}(0,1) = \mathcal{M}_\psi[f(\mathcal{M}_\phi(a_1,b_1;\alpha), a_2), f(\mathcal{M}_\phi(a_1,b_1;\alpha), b_2);\alpha],\\
&\quad \mathcal{F}(t) \le \mathcal{G}(t), \quad t = (t_1,t_2) \in (0,1)^2,\\
&\mathcal{F}(1,1) = \mathcal{G}(1,1) = \psi^{-1}\Big[\alpha^2\psi \circ f(a_1,a_2) + \alpha(1-\alpha)\psi \circ f(a_1,b_2)\\
&\qquad + (1-\alpha)\alpha\psi \circ f(b_1,a_2) + (1-\alpha)^2\psi \circ f(b_1,b_2)\Big].
\end{aligned} \tag{2.1}$$

2. *For* $s = (s_1, s_2) \in (0,1]^2$*, define*

$$\mathcal{I}(s) = \psi^{-1}\left(\frac{\int_0^{s_1}\int_0^{s_2} \psi \circ \mathcal{F}(t_1,t_2) w_1(t_1) w_2(t_2) dt_1 dt_2}{\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2}\right),$$

$$\beta(s) = (\beta_1(s_1), \beta_2(s_2)) = \left(\frac{\int_0^{s_1} t_1 w_1(t_1)dt_1}{\int_0^{s_1} w_1(t_1)dt_1}, \frac{\int_0^{s_2} t_2 w_2(t_2)dt_2}{\int_0^{s_2} w_2(t_2)dt_2}\right),$$

and

$$\mathcal{H}(s) = \psi^{-1}\left(\frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1), t_2) w_2(t_2)dt_2}{2\int_0^{s_2} w_2(t_2)dt_2} + \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1, \beta_2(s_2)) w_1(t_1)dt_1}{2\int_0^{s_1} w_1(t_1)dt_1}\right).$$

Then $\mathcal{F} \circ \beta$*,* $\mathcal{I}$ *and* $\mathcal{G} \circ \beta$ *increase on co-ordinates on* $(0,1]^2$ *and satisfy*

$$\begin{aligned}
&\lim_{s_1 \to 0^+}\lim_{s_2 \to 0^+} \mathcal{F} \circ \beta(s) = \lim_{s_1 \to 0^+}\lim_{s_2 \to 0^+} \mathcal{I}(s) = \lim_{s_1 \to 0^+}\lim_{s_2 \to 0^+} \mathcal{G} \circ \beta(s) = f(\mathcal{M}_\phi(a,b;\alpha)),\\
&\mathcal{F} \circ \beta(s) \le \mathcal{I}(s) \le \mathcal{H}(s) \le \mathcal{G} \circ \beta(s) \le \mathcal{F}(s), \quad s = (s_1,s_2) \in (0,1]^2.
\end{aligned} \tag{2.2}$$

3. *For* $s=(s_1,s_2)\in(0,1]^2$, *define*

$$\mathcal{J}(s)=\psi^{-1}\left(\frac{\int_{s_1}^{1}\int_{s_2}^{1}\psi\circ\mathcal{F}(t_1,t_2)\theta_1(t_1)\theta_2(t_2)dt_1dt_2}{\int_{s_1}^{1}\theta_1(t_1)dt_1\int_{s_2}^{1}\theta_2(t_2)dt_2}\right),$$

$$\gamma(s)=(\gamma_1(s_1),\gamma_2(s_2))=\left(\frac{\int_{s_1}^{1}t_1\theta_1(t_1)dt_1}{\int_{s_1}^{1}\theta_1(t_1)dt_1},\frac{\int_{s_2}^{1}t_2\theta_2(t_2)dt_2}{\int_{0}^{s_2}\theta_2(t_2)dt_2}\right),$$

and

$$\mathcal{K}(s)=\psi^{-1}\left(\frac{\int_{s_2}^{1}\psi\circ\mathcal{G}(\gamma_1(s_1),t_2)w_2(t_2)dt_2}{2\int_{s_2}^{1}w_2(t_2)dt_2}+\frac{\int_{s_1}^{1}\psi\circ\mathcal{G}(t_1,\gamma_2(s_2))w_1(t_1)dt_1}{2\int_{s_1}^{1}w_1(t_1)dt_1}\right).$$

Then $\mathcal{F}\circ\gamma$, $\mathcal{J}$ *and* $\mathcal{G}\circ\gamma$ *increase on co-ordinates on* $[0,1)^2$ *and satisfy*

$$\begin{aligned}&\mathcal{G}(s)\le\mathcal{F}\circ\gamma(s)\le\mathcal{J}(s)\le\mathcal{K}(s)\le\mathcal{G}\circ\gamma(s),\quad s=(s_1,s_2)\in(0,1]^2,\\&\lim_{s_1\to1^-}\lim_{s_2\to1^-}\mathcal{F}\circ\gamma(s)=\lim_{s_1\to1^-}\lim_{s_2\to1^-}\mathcal{J}(s)=\lim_{s_1\to1^-}\lim_{s_2\to1^-}\mathcal{G}\circ\gamma(s)=\mathcal{G}(1,1).\end{aligned}\tag{2.3}$$

If, in addition, $w_i=\theta_i, i=1,2$, *then* $\mathcal{I}(1,1)=\mathcal{J}(0,0)$.

The following three lemmas that will be imperative to the proof of the main result of the above theory.

The first lemma, called Aczél correspondence principle [1] (see also [33, Lemma A.2.2]), reduces the co-ordinates $(\mathcal{M}_\phi,\mathcal{M}_\psi)$-convexity to the usual convexity of a function derived via a change of variable and a change of function.

Lemma 2.2 (Aczél correspondence principle)**.** *If* ψ *is increasing on* J, *then* f *is co-ordinates* $(\mathcal{M}_\phi,\mathcal{M}_\psi)$*-convex on* I *if and only if* $\psi\circ f(\cdot,\phi^{-1}(y))$ *is convex on* $\phi([a_1,b_1])$ *and* $\psi\circ f(\phi^{-1}(x),\cdot)$ *is convex on* $\phi([a_2,b_2])$. *Conversely, if* ψ *is decreasing on* J, *then* f *is co-ordinates* $(\mathcal{M}_\phi,\mathcal{M}_\psi)$*-convex on* I *if and only if* $\psi\circ f(\cdot,\phi^{-1}(y))$ *is concave on* $\phi([a_1,b_1])$ *and* $\psi\circ f(\phi^{-1}(x),\cdot)$ *is concave* $\phi([a_2,b_2])$.

The following lemmas [8] provides a useful inequality related to convex functions, which generalizes the result of Hwang, Tseng and Yang given in [22, Lemma].

Lemma 2.3 ([8])**.** *Let* $H:[A,B]\subset\mathbb{R}\to\mathbb{R}$ *be a convex function and let* $\beta\in[0,1]$. *Then for any* $C,D\in[A,B]$, *with* $\beta A+(1-\beta)B=\beta C+(1-\beta)D$, *one has*

$$\beta H(C)+(1-\beta)H(D)\le\beta H(A)+(1-\beta)H(B).\tag{2.4}$$

Lemma 2.4 ([8])**.** *Let* $P:[0,1]\to\mathbb{R}$ *be continuous and increasing.*

1. *For* $s\in(0,1]$, *define*

$$P_1(s)=\frac{\int_0^s P(t)w_1(t)dt}{\int_0^s w_1(t)dt}.$$

Then P_1 *is increasing on* $(0,1]$, *with*

$$\lim_{s\to0^+}P_1(s)=P(0)\le P_1(s)\le P(s),\quad s\in(0,1].\tag{2.5}$$

2. *Similarly, for* $s \in [0,1)$, *define*

$$P_2(s) = \frac{\int_s^1 P(t)w_2(t)dt}{\int_s^1 w_2(t)dt}.$$

Then P_2 *is increasing on* $[0,1)$, *with*

$$P(s) \le P_2(s) \le P(1) = \lim_{s\to 1^-} P_2(s), \quad s \in [0,1).$$

From the above lemma we can construct the following lemma that it allows one to establish various weighted interpolating inequalities for a continuous and monotonic function on co-ordinates.

Lemma 2.5. *Let* $P : [0,1]^2 \to \mathbb{R}$ *be continuous and increasing on co-ordinates.*

1. *For* $s = (s_1, s_2) \in (0,1]^2$, *define*

$$P_1(s) = \frac{\int_0^{s_1}\int_0^{s_2} P(t_1,t_2)w_1(t_1)w_2(t_2)dt_1dt_2}{\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2}.$$

Then P_1 *is increasing on co-ordinates on* $(0,1]^2$, *with*

$$\lim_{s_1\to 0^+}\lim_{s_2\to 0^+} P_1(s) = P(0,0) \le P_1(s) \le P(s), \quad s = (s_1,s_2) \in (0,1]^2. \tag{2.6}$$

2. *Similarly, for* $s \in [0,1)^2$, *define*

$$P_2(s) = \frac{\int_{s_1}^1\int_{s_2}^1 P(t_1,t_1)\theta_1(t_1)\theta(t_2)dt_1dt_2}{\int_{s_1}^1 \theta_1(t_1)dt_1 \int_{s_2}^1 \theta_2(t_2)dt_2}.$$

Then P_2 *is increasing on co-ordinates on* $[0,1)^2$, *with*

$$P(s) \le P_2(s) \le P(1,1) = \lim_{s_1\to 1^-}\lim_{s_2\to 1^-} P_2(s), \quad s = (s_1,s_2) \in [0,1)^2.$$

We are now in a position to prove the theorem.

Proof of Theorem 2.1. Since ψ is strictly monotonic, we need to examine two possibilities of ψ. Assume first that ψ is strictly increasing on J. But then, because ψ is also continuous on J, ψ^{-1} is continuous and strictly increasing on $\psi(J)$. Furthermore, by Aczél correspondence principle, $\psi \circ f(\cdot, \phi^{-1}(y))$ is convex on $\phi([a_1,b_1])$ and $\psi \circ f(\phi^{-1}(x), \cdot)$ is convex on $\phi([a_2,b_2])$.

1. To show $\mathcal{F}$ is co-ordinates $\mathcal{M}_\psi$-convex on $[0,1]^2$, it suffices to show that $\psi \circ \mathcal{F}$ is co-ordinates convex on $[0,1]^2$. We have

$$\begin{aligned}\psi \circ \mathcal{F}(t) =& \alpha^2\psi \circ f\left(\phi^{-1}(A(t_1)), \phi^{-1}(A(t_2))\right)\\ &+ \alpha(1-\alpha)\psi \circ f\left(\phi^{-1}(A(t_1)), \phi^{-1}(B(t_2))\right)\\ &+ (1-\alpha)\alpha\psi \circ f\left(\phi^{-1}(B(t_1)), \phi^{-1}(A(t_2))\right)\\ &+ (1-\alpha)^2\psi \circ f\left(\phi^{-1}(B(t_1)), \phi^{-1}(B(t_2))\right),\end{aligned}$$

where

$$A(t_i) = t_i\phi(a_i) + (1-t_i)(\alpha\phi(a_i) + (1-\alpha)\phi(b_i)), \quad i = 1, 2, \tag{2.7}$$

and

$$B(t_i) = t_i\phi(b_i) + (1-t_i)(\alpha\phi(a_i) + (1-\alpha)\phi(b_i)), \quad i = 1, 2. \tag{2.8}$$

Since $\psi \circ f \circ \phi^{-1}(t_1, \cdot)$ is convex on $\phi([a_1, b_1])$, $A(t_1)$ and $B(t_1)$ are linear on $[0,1]$, it follows that $\psi \circ \mathcal{F}(t_1, \cdot)$ is convex on $[0,1]$ as claimed. Similarly, $\psi \circ \mathcal{F}(\cdot, t_2)$ is convex on $[0,1]$. The co-ordinates $\mathcal{M}_\psi$-convexity of $\mathcal{G}$ on $[0,1]^2$ immediately follows from the definition of $\mathcal{G}$.

Next, it is easily seen that

$$\begin{aligned}
&\mathcal{F}(1,1) = \mathcal{G}(1,1),\\
&\mathcal{F}(0,0) = \mathcal{G}(0,0) = f(\mathcal{M}_\phi(a,b;\alpha)),\\
&\mathcal{F}(1,0) = \mathcal{G}(1,0) = \mathcal{M}_\psi[f(a_1, \mathcal{M}_\phi(a_2,b_2;\alpha)), f(b_1, \mathcal{M}_\phi(a_2,b_2;\alpha))],\\
&\mathcal{F}(0,1) = \mathcal{F}(0,1) = \mathcal{M}_\psi[f(\mathcal{M}_\phi(a_1,b_1;\alpha), a_2), f(\mathcal{M}_\phi(a_1,b_1;\alpha), b_2)].
\end{aligned}$$

Now, by the convexity of $\psi \circ f(\phi^{-1}(x), \cdot)$ and $\psi \circ f(\cdot, \phi^{-1}(y))$,

$$\begin{aligned}
&\psi \circ f(\phi^{-1}(A(t_1)), \phi^{-1}(A(t_2)))\\
&\le t_1\psi \circ f(a_1, \phi^{-1}(A(t_2)) + (1-t_1)\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), \phi^{-1}(A(t_2))\\
&\le t_1\Big(t_2\psi \circ f(a_1,a_2) + (1-t_2)\psi \circ f(a_1, \mathcal{M}_\phi(a_2,b_2;\alpha))\Big)\\
&\quad + (1-t_1)\Big(t_2\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), a_2) + (1-t_2)\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), \mathcal{M}_\phi(a_2,b_2;\alpha))\Big),
\end{aligned}$$

$$\begin{aligned}
&\psi \circ f(\phi^{-1}(A(t_1)), \phi^{-1}(B(t_2)))\\
&\le t_1\psi \circ f(a_1, \phi^{-1}(B(t_2)) + (1-t_1)\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), \phi^{-1}(B(t_2))\\
&\le t_1\Big(t_2\psi \circ f(a_1,b_2) + (1-t_2)\psi \circ f(a_1, \mathcal{M}_\phi(a_2,b_2;\alpha))\Big)\\
&\quad + (1-t_1)\Big(t_2\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), b_2) + (1-t_2)\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), \mathcal{M}_\phi(a_2,b_2;\alpha))\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\psi \circ f(\phi^{-1}(B(t_1)), \phi^{-1}(A(t_2)))\\
&\le t_1\psi \circ f(b_1, A(t_2)) + (1-t_1)\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), A(t_2))\\
&\le t_1\Big(t_2\psi \circ f(b_1,a_2) + (1-t_2)\psi \circ f(b_1, \mathcal{M}_\phi(a_2,b_2;\alpha))\Big)\\
&\quad + (1-t_1)\Big(t_2\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), a_2) + (1-t_2)\psi \circ f(\mathcal{M}_\phi(a_1,b_1;\alpha), \mathcal{M}_\phi(a_2,b_2;\alpha))\Big),
\end{aligned}$$

$$\psi \circ f(\phi^{-1}(B(t_1)), \phi^{-1}(B(t_2)))$$
$$\le t_1\psi \circ f(b_1, B(t_2)) + (1-t_1)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), B(t_2))$$
$$\le t_1\Big(t_2\psi \circ f(b_1, b_2) + (1-t_2)\psi \circ f(b_1, \mathcal{M}_\phi(a_2, b_2; \alpha))\Big)$$
$$+ (1-t_1)\Big(t_2\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), b_2) + (1-t_2)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha))\Big).$$

We thus get

$$\begin{aligned}\psi \circ \mathcal{F}(t) \le t_1t_2\Big(&\alpha^2\psi \circ f(a_1, a_2) + \alpha(1-\alpha)\psi \circ f(a_1, b_2)\\ &+ (1-\alpha)\alpha\psi \circ f(b_1, a_2) + (1-\alpha)^2\psi \circ f(b_1, b_2)\Big)\\ &+ t_1(1-t_2)\Big(\alpha\psi \circ f(a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)) + (1-\alpha)\psi \circ f(b_1, \mathcal{M}_\phi(a_2, b_2; \alpha))\Big)\\ &+ t_2(1-t_1)\Big(\alpha\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), a_2) + (1-\alpha)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), b_2)\Big)\\ &+ (1-t_1)(1-t_2)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha))\\ =\ &\psi \circ \mathcal{G}(t)\end{aligned}$$

and, because ψ^{-1} is increasing on $\psi(J)$,

$$\mathcal{F}(t) \le \mathcal{G}(t), \quad t = (t_1, t_2) \in (0, 1]^2,$$

whence (2.1) is verified.

We proceed to show that $\mathcal{F}$ is increasing co-ordinates on $[0, 1]^2$. To this end, suppose that $0 < t_i < r_i \le 1, i = 1, 2$. By the co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convexity of f,

$$\begin{aligned}\mathcal{F}(0, t_2) &= \psi^{-1}\Big[\alpha^2\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{L}_2(t_2)) + \alpha(1-\alpha)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{R}_2(t_2))\\ &\qquad + (1-\alpha)\alpha\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{L}_2(t_2)) + (1-\alpha)^2\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{R}_2(t_2))\Big]\\ &= \psi^{-1}\Big[\alpha\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{L}_2(t_2)) + (1-\alpha)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{R}_2(t_2))\Big]\\ &\le \psi^{-1}\Big[\alpha\psi \circ \mathcal{M}_\psi[f(\mathcal{L}_1(t_1)), \mathcal{L}_2(t_2)), f(\mathcal{R}_1(t_1)), \mathcal{L}_2(t_2)); \alpha]\\ &\qquad + (1-\alpha)\psi \circ \mathcal{M}_\psi[f(\mathcal{L}_1(t_1)), \mathcal{L}_2(t_2)), f(\mathcal{R}_1(t_1)), \mathcal{R}_2(t_2)); \alpha]\Big]\\ &= \mathcal{F}(t_1, t_2)\end{aligned}$$

which yields

$$\psi \circ \mathcal{F}(t_1, t_2) \ge \psi \circ \mathcal{F}(0, t_2).$$

Similarly,

$$\psi \circ \mathcal{F}(t_1, t_2) \geq \psi \circ \mathcal{F}(t_1, 0).$$

Together with the co-ordinates convexity of $\psi \circ \mathcal{F}$, this gives

$$\frac{\psi \circ \mathcal{F}(r_1, t_2) - \psi \circ \mathcal{F}(t_1, t_2)}{r_1 - t_1} \geq \frac{\psi \circ \mathcal{F}(t_1, t_2) - \psi \circ \mathcal{F}(0, t_2)}{t_1 - 0} \geq 0$$

and

$$\frac{\psi \circ \mathcal{F}(t_1, r_2) - \psi \circ \mathcal{F}(t_1, t_2)}{r_2 - t_2} \geq \frac{\psi \circ \mathcal{F}(t_1, t_2) - \psi \circ \mathcal{F}(t_1, 0)}{t_2 - 0} \geq 0,$$

which implies that $\psi \circ \mathcal{F}$ is increasing co-ordinates on $[0,1]^2$. Since ψ^{-1} is increasing on $\psi(J)$, we conclude that $\mathcal{F}$ is increasing co-ordinates on $[0,1]^2$ as desired. Since

$$\begin{aligned} \psi \circ \mathcal{G}(t) = t_1 \Big(& t_2 \psi \circ \mathcal{F}(1,1) + (1-t_2)\psi \circ \mathcal{F}(1,0) \Big) \\ & + (1-t_1)\Big(t_2 \psi \circ \mathcal{F}(0,1) + (1-t_2)\psi \circ \mathcal{F}(0,0) \Big), \end{aligned}$$

$$\begin{aligned} \psi \circ \mathcal{G}(t) = t_2 \Big(& t_1 \psi \circ \mathcal{F}(1,1) + (1-t_1)\psi \circ \mathcal{F}(0,1) \Big) \\ & + (1-t_2)\Big(t_1 \psi \circ \mathcal{F}(1,0) + (1-t_1)\psi \circ \mathcal{F}(0,0) \Big) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}(0,1) \leq \mathcal{F}(1,1), \quad \mathcal{F}(1,0) \leq \mathcal{F}(1,1), \\ \mathcal{F}(0,0) \leq \mathcal{F}(0,1), \quad \mathcal{F}(0,0) \leq \mathcal{F}(1,0) \end{aligned}$$

it follows that $\psi \circ \mathcal{G}$, and so does $\mathcal{G}$, increases co-ordinates on $[0,1]^2$.

2. Applying Lemma 2.4 for $P = \psi \circ \mathcal{F}$, we conclude that $\psi \circ \mathcal{I}$ is increasing co-ordinates on $(0,1]^2$, with

$$\lim_{s_1 \to 0^+} \lim_{s_2 \to 0^+} \psi \circ \mathcal{I}(s) = \psi \circ \mathcal{F}(0,0) = \psi \circ f(\mathcal{M}_\phi(a,b;\alpha)).$$

Since ψ^{-1} is continuous and strictly increasing on $\psi(J)$, it follows that $\mathcal{I}$ is increasing co-ordinates on $(0,1]^2$ and

$$\lim_{s_1 \to 0^+} \lim_{s_2 \to 0^+} \mathcal{I}(s) = f(\mathcal{M}_\phi(a,b;\alpha)).$$

Again, by Lemma 2.4, β is increasing co-ordinates on $(0,1]^2$, with

$$\lim_{s_1 \to 0^+} \lim_{s_2 \to 0^+} \beta(s) = (0,0) \leq \beta(s) \leq s, \quad s \in (0,1]^2.$$

Thus, the first part of the theorem asserts that $\mathcal{F} \circ \beta$ and $\mathcal{G} \circ \beta$ are well-defined, increasing co-ordinates on $(0,1]^2$ and

$$\lim_{s_1 \to 0^+} \lim_{s_2 \to 0^+} \mathcal{F} \circ \beta(s) = \lim_{s_1 \to 0^+} \lim_{s_2 \to 0^+} \mathcal{G} \circ \beta(s) = f(\mathcal{M}_\phi(a,b;\alpha)).$$

Our next goal is to show the inequalities in (2.2). Fix $s = (s_1, s_2) \in (0,1]^2$. Applying Jensen's inequality (see, for example, [38, Chapter 2]) to the convex function $\psi \circ \mathcal{F}_{t_i}(t_j)$ on the interval $[0, s_j]$ with respect to the measure $w_j(t_j)dt_j$, $i, j = 1, 2$ we obtain

$$\psi \circ \mathcal{F}\left(\frac{\int_0^{s_1} t_1 w_1(t_1)dt_1}{\int_0^{s_1} w_1(t_1)dt_1}, \frac{\int_0^{s_2} t_2 w_2(t_2)dt_2}{\int_0^{s_2} w_2(t_2)dt_2}\right) \leq \frac{\int_0^{s_1}\int_0^{s_2} \psi \circ \mathcal{F}(t_1,t_2)w_1(t_1)w_2(t_2)dt_1dt_2}{\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2},$$

which yields

$$\mathcal{F} \circ \beta(s) \leq \mathcal{I}(s).$$

Sine $\mathcal{F}(t_1, t_2) \leq \mathcal{G}(t_1, t_2)$ for all $t_2 \in [0, s_2]$ we get that

$$\frac{\int_0^{s_1} \psi \circ \mathcal{F}(t_1,t_2)w_1(t_1)dt_1}{\int_0^{s_1} w_1(t_1)dt_1} \leq \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1,t_2)w_1(t_1)dt_1}{\int_0^{s_1} w_1(t_1)dt_1} \leq \psi \circ \mathcal{G}(\beta_1(s_1), t_2).$$

Hence

$$\frac{\int_0^{s_1}\int_0^{s_2} \psi \circ \mathcal{F}(t_1,t_2)w_1(t_1)w_2(t_2)dt_1dt_2}{\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2} \leq \frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1),t_2)w_2(t_2)dt_2}{\int_0^{s_2} w_2(t_2)dt_2}. \tag{2.9}$$

In a similar way we get

$$\frac{\int_0^{s_1}\int_0^{s_2} \psi \circ \mathcal{F}(t_1,t_2)w_1(t_1)w_2(t_2)dt_1dt_2}{\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2} \leq \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1,\beta_2(s_2))w_1(t_1)dt_1}{\int_0^{s_1} w_1(t_1)dt_1}. \tag{2.10}$$

Summing (2.9) and (2.10) we obtain

$$\begin{aligned}&\frac{\int_0^{s_1}\int_0^{s_2} \psi \circ \mathcal{F}(t_1,t_2)w_1(t_1)w_2(t_2)dt_1dt_2}{\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2} \\ &\leq \frac{1}{2}\left(\frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1),t_2)w_2(t_2)dt_2}{\int_0^{s_2} w_2(t_2)dt_2} + \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1,\beta_2(s_2))w_1(t_1)dt_1}{\int_0^{s_1} w_1(t_1)dt_1}\right),\end{aligned}$$

which, as the function ψ^{-1} is increasing, implies

$$\mathcal{I}(s) \leq \psi^{-1}\left(\frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1),t_2)w_2(t_2)dt_2}{2\int_0^{s_2} w_2(t_2)dt_2} + \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1,\beta_2(s_2))w_1(t_1)dt_1}{2\int_0^{s_1} w_1(t_1)dt_1}\right) = \mathcal{H}(s).$$

Next, it is easily seen that

$$\mathcal{H}(s) \leq \mathcal{G} \circ \beta(s).$$

It remains to show

$$\mathcal{G} \circ \beta(s) \leq \mathcal{F}(s).$$

We utilize Lemma 2.3, with $H = H(x,y) = \psi \circ f(\phi^{-1}(x), \phi^{-1}(y))$, $A = \min\{A(s_1), B(s_1)\}$, $B = \max\{A(s_1), B(s_1)\}$, $C = \min\{A(\beta_1(s_1)), B(\beta_1(s_1))\}$, $D = \max\{A(\beta_1(s_1)), B(\beta_1(s_1))\}$, and

$$\beta = \begin{cases} \alpha & \text{if} \quad A(s_1) \leq B(s_1), \\ 1-\alpha & \text{if} \quad A(s_1) > B(s_1), \end{cases}$$

where $A(\cdot)$ and $B(\cdot)$ are as in (2.7) and (2.8), respectively. To do this, we need to ensure that $C, D \in [A, B]$, with $\beta A + (1-\beta)B = \beta C + (1-\beta)D$. But this immediately follows from the fact that

$$\beta A + (1-\beta)B = \beta C + (1-\beta)D = \alpha\phi(a) + (1-\alpha)\phi(b)$$

and

$$B - A = s_1|\phi(b) - \phi(a)| \geq \beta_1(s_1)|\phi(b) - \phi(a)| = D - C.$$

A computation shows that

$$\psi \circ \mathcal{G}(\beta_1(s_1), s_2) = \beta H(C, s_2) + (1-\beta)H(D, s_2)$$

and

$$\psi \circ \mathcal{F}(s_1, s_2) = \beta H(A, s_2) + (1-\beta)H(B, s_2).$$

On account of (2.4), we have

$$\psi \circ \mathcal{G}(\beta_1(s_1), s_2) \leq \psi \circ \mathcal{F}(s_1, s_2),$$

therefore

$$\psi \circ \mathcal{G}(\beta_1(s_1), \beta_2(s_2)) \leq \psi \circ \mathcal{G}(\beta_1(s_1), s_2) \leq \psi \circ \mathcal{F}(s_1, s_2)$$

which establishes the desired inequality.

3. We proceed similarly as in the proof of part 2, with β and $(0, s_1] \times (0, s_2]$, respectively, replaced by γ and $[s_1, 1) \times [s_2, 1)$, we can assert that $\mathcal{F} \circ \gamma$, $\mathcal{J}$ and $\mathcal{G} \circ \gamma$ increase co-ordinates on $[0, 1)^2$ and (2.3) follows. If $w_i = \theta_i, i = 1, 2$, then $\mathcal{I}(1, 1) = \mathcal{J}(0, 0)$, which is clear from the definitions of $\mathcal{I}$ and $\mathcal{J}$. □

Let us now mention another important consequence of Theorem 2.1. It should be pointed out that a variety of Fejér type inequalities for co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$-convex functions can be produced by choosing various weights, w_1, w_2 and θ_1, θ_2. For instance, let us choose

$$w_i(t_i) = (1-\alpha)g_i(t_i) \circ \mathcal{L}_i(t_i) + \alpha g_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [0, 1], i = 1, 2$$

and

$$\theta_i(t_i) = (1-\alpha)h_i(t_i) \circ \mathcal{L}_i(t_i) + \alpha h_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [0, 1], i = 1, 2$$

where $g_i, h_i : I \to [0, \infty)$ are given to satisfy

$$\frac{1-\alpha}{\alpha} g_i(t_i) \circ \mathcal{L}_i(t_i) = \frac{\alpha}{1-\alpha} g_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [0, s_i] \tag{2.11}$$

and

$$\frac{1-\alpha}{\alpha} h_i(t_i) \circ \mathcal{L}_i(t_i) = \frac{\alpha}{1-\alpha} h_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [s_i, 1]. \tag{2.12}$$

Notice that if $\alpha = 1/2$ and $\phi(x) = x$, then the assumptions (2.11) and (2.12) reduce to the ones that g_i and h_i are symmetric to $(a_i + b_i)/2$ for $i = 1, 2$.

A computation, using (2.11) and $\mathcal{L}(0,0)=\mathcal{R}(0,0)$, forces

$$\begin{aligned}
&\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2 \\
&= \left((1-\alpha)\int_0^{s_1} g_1(t_1)\circ\mathcal{L}_1(t_1)dt_1 + \alpha\int_0^{s_1} g_1(t_1)\circ\mathcal{R}_1(t_1)dt_1\right) \\
&\qquad \times \left((1-\alpha)\int_0^{s_2} g_2(t_2)\circ\mathcal{L}_2(t_2)dt_2 + \alpha\int_0^{s_2} g_2(t_2)\circ\mathcal{R}_2(t_2)dt_2\right) \\
&= \frac{1}{(\phi(b_1)-\phi(a_1))(\phi(b_2)-\phi(a_2))}\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} g_1(x)d\phi(x)\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} g_2(x)d\phi(x),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{s_1}\int_0^{s_2} \psi\circ\mathcal{F}(t_1,t_2)w_1(t_1)w_2(t_2)dt_1dt_2 \\
&= \alpha\int_0^{s_1}\left(\int_0^{s_2}[\alpha\psi\circ f(\mathcal{L}_1(t_1),\mathcal{L}_2(t_2)) + (1-\alpha)\psi\circ f(\mathcal{L}_1(t_1),\mathcal{R}_2(t_2))]w_2(t_2)dt_2\right)w_1(t_1)dt_1 \\
&\qquad + (1-\alpha)\int_0^{s_1}\left(\int_0^{s_2}[\alpha\psi\circ f(\mathcal{R}_1(t_1),\mathcal{L}_2(t_2)) + (1-\alpha)\psi\circ f(\mathcal{R}_1(t_1),\mathcal{R}_2(t_2))]w_2(t_2)dt_2\right)w_1(t_1)dt_1 \\
&= \frac{\alpha}{\phi(b_2)-\phi(a_2)}\int_0^{s_1}\left(\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)}\psi\circ f(\mathcal{L}_1(t_1),x_2)g_2(x_2)d\phi(x_2)\right)w_1(t_1)dt_1 \\
&\qquad + \frac{1-\alpha}{\phi(b_2)-\phi(a_2)}\int_0^{s_1}\left(\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)}\psi\circ f(\mathcal{R}_1(t_1),x_2)g_2(x_2)d\phi(x_2)\right)w_1(t_1)dt_1 \\
&= \frac{1}{\phi(b_2)-\phi(a_2)}\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)}\left(\int_0^{s_1}[\alpha\psi\circ f(\mathcal{L}_1(t_1),x_2) + (1-\alpha)\psi\circ f(\mathcal{R}_1(t_1),x_2)]w_1(t_1)dt_1\right)g_2(x_2)d\phi(x_2) \\
&= \frac{1}{(\phi(b_1)-\phi(a_1))(\phi(b_2)-\phi(a_2))}\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)}\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)}\psi\circ f(x_1,x_2)g_1(x_1)g_2(x_2)d\phi(x_1)d\phi(x_2)
\end{aligned}$$

and hence

$$\mathcal{I}(s_1,s_2) = \psi^{-1}\left(\frac{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)}\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)}\psi\circ f(x_1,x_2)g_1(x_1)g_2(x_2)d\phi(x_1)d\phi(x_2)}{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} g_1(x_1)d\phi(x_1)\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} g_2(x_2)d\phi(x_2)}\right).$$

Similarly, by (2.12), $\mathcal{L}(1,1)=(a_1,a_2)$ and $\mathcal{R}(1,1)=(b_1,b_2)$,

$$\mathcal{J}(s_1,s_2) = \psi^{-1}\left(\frac{\mathcal{J}_1(s_1,s_2)}{\mathcal{J}_2(s_1,s_2)}\right),$$

where

$$\begin{aligned}\mathcal{J}_1(s_1,s_2) = &\int_{a_1}^{\mathcal{L}_1(s_1)}\int_{a_2}^{\mathcal{L}_2(s_2)} \psi\circ f(x_1,x_2)h_1(x_1)h_2(x_2)d\phi(x_1)d\phi(x_2)\\ &+\int_{a_1}^{\mathcal{L}_1(s_1)}\int_{\mathcal{R}_2(s_2)}^{b_2} \psi\circ f(x_1,x_2)h_1(x_1)h_2(x_2)d\phi(x_1)d\phi(x_2)\\ &+\int_{\mathcal{R}_1(s_1)}^{b_1}\int_{a_2}^{\mathcal{L}_2(s_2)} \psi\circ f(x_1,x_2)h_1(x_1)h_2(x_2)d\phi(x_1)d\phi(x_2)\\ &+\int_{\mathcal{R}_1(s_1)}^{b_1}\int_{\mathcal{R}_2(s_2)}^{b_2} \psi\circ f(x_1,x_2)h_1(x_1)h_2(x_2)d\phi(x_1)d\phi(x_2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_2(s_1,s_2) = &\left(\int_{a_1}^{\mathcal{L}_1(s_1)} h_1(x_1)d\phi(x_1) + \int_{\mathcal{R}_1(s_1)}^{b} h_1(x_1)d\phi(x_1)\right)\\ &\times\left(\int_{a_2}^{\mathcal{L}_2(s_2)} h_2(x_2)d\phi(x_2) + \int_{\mathcal{R}_2(s_2)}^{b} h_2(x_2)d\phi(x_2)\right).\end{aligned}$$

Together with the aid of Theorem 2.1, we establish the following corollary.

Corollary 2.6. *Suppose that $g_i, h_i : [a_i, b_i] \to [0,\infty)$ are integrable for $i = 1, 2$, with $\int_0^{s_i} g_i\circ\mathcal{L}_i(t_i)dt_i > 0$ and $\int_{s_i}^1 h_i\circ\mathcal{R}_i(t_i)dt_i > 0$ for all $s_i \in (0,1)$, and satisfy (2.11) and (2.12). Then, for $s = (s_1, s_2) \in (0,1)^2$,*

$$\begin{aligned}f(\mathcal{M}_\phi(a,b;\alpha)) &\le \mathcal{F}\left(\frac{\int_0^{s_1} t_1 g_1\circ\mathcal{L}_1(t_1)dt_1}{\int_0^{s_1} g_1\circ\mathcal{L}_1(t_1)dt_1}, \frac{\int_0^{s_2} t_2 g_2\circ\mathcal{L}_2(t_2)dt_2}{\int_0^{s_2} g_2\circ\mathcal{L}_2(t_2)dt_2}\right)\\ &\le \psi^{-1}\left(\frac{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)}\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} \psi\circ f(x_1,x_2)g_1(x_1)g_2(x_2)d\phi(x_1)d\phi(x_2)}{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} g_1(x_1)d\phi(x_1)\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} g_2(x_2)d\phi(x_2)}\right)\\ &\le \mathcal{G}\left(\frac{\int_0^{s_1} t_1 g_1\circ\mathcal{L}_1(t_1)dt_1}{\int_0^{s_1} g_1\circ\mathcal{L}_1(t_1)dt_1}, \frac{\int_0^{s_2} t_2 g_2\circ\mathcal{L}_2(t_2)dt_2}{\int_0^{s_2} g_2\circ\mathcal{L}_2(t_2)dt_2}\right)\\ &\le \mathcal{F}(s) \le \mathcal{H}(s) \le \mathcal{G}(s) \qquad (2.13)\\ &\le \mathcal{F}\left(\frac{\int_{s_1}^1 t_1 h_1\circ\mathcal{L}_1(t_1)dt_1}{\int_{s_1}^1 h_1\circ\mathcal{L}_1(t_1)dt_1}, \frac{\int_{s_2}^1 t_2 h_2\circ\mathcal{L}_2(t_2)dt_2}{\int_{s_2}^1 h_2\circ\mathcal{L}_2(t_2)dt_2}\right)\\ &\le \psi^{-1}\left(\frac{\mathcal{J}_1(s_1,s_2)}{\mathcal{J}_2(s_1,s_2)}\right)\\ &\le \mathcal{G}\left(\frac{\int_{s_1}^1 t_1 h_1\circ\mathcal{L}_1(t_1)dt_1}{\int_{s_1}^1 h_1\circ\mathcal{L}_1(t_1)dt_1}, \frac{\int_{s_2}^1 t_2 h_2\circ\mathcal{L}_2(t_2)dt_2}{\int_{s_2}^1 h_2\circ\mathcal{L}_2(t_2)dt_2}\right) \le \mathcal{G}(1,1).\end{aligned}$$

Remark 2.7. It turns out that a great deal of existing inequalities of Hermite-Hadamard and Fejér type for different kinds of convexity can be attained from Corollary 2.6.

1. Let us consider $\phi(x) = x$ and $\psi(x) = x$. If $\alpha = 1/2$ then (2.13) offers a refinement of the inequalities due to Farid, Marwan and Atiq Ur Rehman [15].
2. Next, if we choose $\phi(x) = x$ and $\psi(x) = x$. If $\alpha = 1/2$, $g_1 = g_2 = h_1 = h_2 = 1$ and $s_1 = s_2 = 1/2$, (2.13) implies

$$\begin{aligned}
& f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) \\
& \leq \frac{1}{4}\left[f\left(\frac{5a_1+3b_1}{8}, \frac{5a_2+3b_2}{8}\right) + f\left(\frac{5a_1+3b_1}{8}, \frac{3a_2+5b_2}{8}\right)\right. \\
& \qquad \left. + f\left(\frac{3a_1+5b_1}{8}, \frac{5a_2+3b_2}{8}\right) + f\left(\frac{3a_1+5b_1}{8}, \frac{3a_2+5b_2}{8}\right)\right] \\
& \leq \frac{4}{(b_1-a_1)(b_2-a_2)} \int_{(3a_1+b_1)/4}^{(a_1+3b_1)/4} \int_{(3a_2+b_2)/4}^{(a_2+3b_2)/4} f(x_1,x_2)dx_1dx_2 \\
& \leq \frac{1}{16}\left[\frac{f(a_1,a_2)+f(a_1,b_2)+f(b_1,a_2)+f(b_1,b_2)}{4} + 3\frac{f(a_1,\frac{a_2+b_2}{2}) + f(b_1,\frac{a_2+b_2}{2})}{2}\right. \\
& \qquad \left. + 3\frac{f(\frac{a_1+b_1}{2},a_2) + f(\frac{a_1+b_1}{2},b_2)}{2} + f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)\right] \\
& \leq \frac{1}{4}\left[f\left(\frac{3a_1+b_1}{4}, \frac{3a_2+b_2}{4}\right) + f\left(\frac{3a_1+b_1}{4}, \frac{a_2+3b_2}{4}\right)\right. \\
& \qquad \left. + f\left(\frac{a_1+3b_1}{4}, \frac{3a_2+b_2}{4}\right) + f\left(\frac{a_1+3b_1}{4}, \frac{a_2+3b_2}{4}\right)\right] \\
& \leq \frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1,x_2)dx_1dx_2 \\
& \leq \frac{1}{8}\left[\frac{1}{b_1-a_1} \int_{a_1}^{b_1} \left(f(x_1,a_2) + f(x_1,b_2) + 2f\left(x_1, \frac{a_2+b_2}{2}\right)\right)dx_1\right. \\
& \qquad \left. + \frac{1}{b_2-a_2} \int_{a_2}^{b_2} \left(f(a_1,x_2) + f(b_1,x_2) + 2f\left(\frac{a_1+b_1}{2}, x_2\right)\right)dx_2\right] \\
& \leq \frac{1}{4}\left[\frac{f(a_1,a_2)+f(a_1,b_2)+f(b_1,a_2)+f(b_1,b_2)}{4} + \frac{f(a_1,\frac{a_2+b_2}{2}) + f(b_1,\frac{a_2+b_2}{2})}{2}\right. \\
& \qquad \left. + \frac{f(\frac{a_1+b_1}{2},a_2) + f(\frac{a_1+b_1}{2},b_2)}{2} + f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)\right] \\
& \leq \frac{f(a_1,a_2)+f(a_1,b_2)+f(b_1,a_2)+f(b_1,b_2)}{4}
\end{aligned} \tag{2.14}$$

which offers a refinement of (2.13) and ones due to Bakula [3], Özdemir [37].

3. Moreover, if $f(x_1,x_2)=f(x_1)g(x_2)$ then (2.13) implies

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&\le \frac{1}{4}\Bigg[f\left(\frac{5a+3b}{8}\right)g\left(\frac{5a+3b}{8}\right)+f\left(\frac{5a+3b}{8}\right)g\left(\frac{3a+5b}{8}\right)\\
&\qquad +f\left(\frac{3a+5b}{8}\right)g\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)g\left(\frac{3a+5b}{8}\right)\Bigg]\\
&\le \frac{2}{(b-a)}\int_{(3a+b)/4}^{(a+3b)/4} f(x)g(x)dx\\
&\le \frac{1}{16}\Bigg[\frac{f(a)g(a)+f(a)g(b)+f(b)g(a)+f(b)g(b)}{4}+3\frac{f(a)g(\frac{a+b}{2})+f(b)g(\frac{a+b}{2})}{2}\\
&\qquad +3\frac{f(\frac{a+b}{2})g(a)+f(\frac{a+b}{2})g(b)}{2}+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\Bigg]\\
&\le \frac{1}{4}\Bigg[f\left(\frac{3a+b}{4}\right)g\left(\frac{3a+b}{4}\right)+f\left(\frac{3a+b}{4}\right)g\left(\frac{a+3b}{4}\right)\\
&\qquad +f\left(\frac{a+3b}{4}\right)g\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)g\left(\frac{a+3b}{4}\right)\Bigg]\\
&\le \frac{1}{(b-a)}\int_a^b f(x)g(x)dx\\
&\le \frac{1}{8(b-a)}\Bigg[\left(g(a)+g(b)+2g\left(\frac{a+b}{2}\right)\right)\int_a^b f(x)dx\\
&\qquad +\left(f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right)\int_a^b g(x)dx\Bigg]\\
&\le \frac{1}{4}\Bigg[\frac{f(a)g(a)+f(a)g(b)+f(b)g(a)+f(b)g(b)}{4}+\frac{f(a)g(\frac{a+b}{2})+f(b)g(\frac{a+b}{2})}{2}\\
&\qquad +\frac{f(\frac{a+b}{2})g(a)+f(\frac{a+b}{2})g(b)}{2}+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\Bigg]\\
&\le \frac{f(a)g(a)+f(a)g(b)+f(b)g(a)+f(b)g(b)}{4}.
\end{aligned} \tag{2.15}$$

Let us consider kernels, say $K_i:\phi_i(I)\times\phi_i(I)\to[0,\infty)$, for $i=1,2$ and define

$$\mathcal{K}^{\phi_1,\phi_2}_{a_1+,a_2+}[f](x_1,x_2)=\int_{a_1}^{x_1}\int_{a_2}^{x_2} K_1(\phi_1(x_1),\phi_1(y_1))K_2(\phi_2(x_2),\phi_2(y_2))f(y_1,y_2)d\phi(y_1)d\phi(y_2) \tag{2.16}$$

for $x_1 > a_1, x_2 > a_2$,

$$\mathcal{K}^{\phi_1,\phi_2}_{a_1+,b_2-}[f](x_1,x_2) = \int_{a_1}^{x_1}\int_{x_2}^{b_2} K_1(\phi_1(x_1),\phi_1(y_1))K_2(\phi_2(x_2),\phi_2(y_2))f(y_1,y_2)d\phi(y_1)d\phi(y_2) \tag{2.17}$$

for $x_1 > a_1, x_2 < b_2$,

$$\mathcal{K}^{\phi_1,\phi_2}_{b_1-,a_2+}[f](x_1,x_2) = \int_{x_1}^{b_1}\int_{a_2}^{x_2} K_1(\phi_1(x_1),\phi_1(y_1))K_2(\phi_2(x_2),\phi_2(y_2))f(y_1,y_2)d\phi(y_1)d\phi(y_2) \tag{2.18}$$

for $x_1 < b_1, x_2 > a_2$,

$$\mathcal{K}^{\phi_1,\phi_2}_{b_1-,b_2-}[f](x_1,x_2) = \int_{x_1}^{b_1}\int_{x_2}^{b_2} K_1(\phi_1(x_1),\phi_1(y_1))K_2(\phi_2(x_2),\phi_2(y_2))f(y_1,y_2)d\phi(y_1)d\phi(y_2) \tag{2.19}$$

for $x_1 < b_1, x_2 < b_2$, as long as the integrals exist and are finite.

Remark 2.8. We emphasize that our definition agrees with many known fractional integrals existing in the literature as special cases.

1. Let us first consider

$$K_i(u,v) = \frac{1}{\Gamma(\nu_i)}|u-v|^{\nu_i-1}, \quad u,v \in \phi_i(I), i = 1,2,$$

where $\nu_i > 0$. Then integral operators (2.16)–(2.19) become the fractional integrals of a function with respect to another function known:

$$\mathcal{I}^{\phi_1,\phi_2}_{a_1+,a_2+}[f](x_1,x_2) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{a_1}^{x_1}\int_{a_2}^{x_2} |\phi_1(x_1)-\phi_1(y_1)|^{\nu_1-1}|\phi_2(x_2)-\phi_2(y_2)|^{\nu_2-1}f(y_1,y_2)d\phi(y_1)d\phi(y_2)$$

for $x_1 > a_1, x_2 > a_2$,

$$\mathcal{I}^{\phi_1,\phi_2}_{a_1+,b_2-}[f](x_1,x_2) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{a_1}^{x_1}\int_{x_2}^{b_2} |\phi_1(x_1)-\phi_1(y_1)|^{\nu_1-1}|\phi_2(x_2)-\phi_2(y_2)|^{\nu_2-1}f(y_1,y_2)d\phi(y_1)d\phi(y_2)$$

for $x_1 > a_1, x_2 < b_2$,

$$\mathcal{I}^{\phi_1,\phi_2}_{b_1-,a_2+}[f](x_1,x_2) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{x_1}^{b_1}\int_{a_2}^{x_2} |\phi_1(x_1)-\phi_1(y_1)|^{\nu_1-1}|\phi_2(x_2)-\phi_2(y_2)|^{\nu_2-1}f(y_1,y_2)d\phi(y_1)d\phi(y_2)$$

for $x_1 < b_1, x_2 > a_2$,

$$\mathcal{I}^{\phi_1,\phi_2}_{b_1-,b_2-}[f](x_1,x_2) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{x_1}^{b_1}\int_{x_2}^{b_2} |\phi_1(x_1)-\phi_1(y_1)|^{\nu_1-1}|\phi_2(x_2)-\phi_2(y_2)|^{\nu_2-1}f(y_1,y_2)d\phi(y_1)d\phi(y_2)$$

for $x_1 < b_1, x_2 < b_2$,
These operators include the Riemann-Liouville fractional integral [41], which the choice $\phi(x) = x$.

In Corollary 2.6, for $s_i \in (0,1), i = 1,2$, let us choose

$$g_i(x) = [K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i(x_i)) + K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i(x_i))]\, u_i(x_i), \quad x_i \in [\mathcal{L}_i(s_i), \mathcal{R}_i(s_i)], i = 1,2$$

and

$$h_i(x_i) = \begin{cases} K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i(x_i))v_i(x_i) & \text{if} \quad x_i \in [a_i, \mathcal{L}_i(s_i)], \\ K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i(x_i))v_i(x_i) & \text{if} \quad x_i \in [\mathcal{R}_i(s_i), b_i], \end{cases}$$

where $u_i, v_i : [a_i, b_i] \to [0, \infty)$, for $i = 1,2$, are given in such a way that the assumptions (2.11) and (2.12) are guaranteed, i.e.,

$$\begin{aligned} &\frac{1-\alpha}{\alpha}[K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i \circ \mathcal{L}_i(t_i)) + K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i \circ \mathcal{L}_i(t_i))]u_i \circ \mathcal{L}_i(t_i) \\ =&\frac{\alpha}{1-\alpha}[K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i \circ \mathcal{R}_i(t_i)) + K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi \circ \mathcal{R}_i(t_i))]u_i \circ \mathcal{R}_i(t_i), \end{aligned} \tag{2.20}$$

for $t_i \in [0, s_i], i = 1,2$ and

$$\frac{1-\alpha}{\alpha}K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i \circ \mathcal{L}_i(t_i))v_i \circ \mathcal{L}_i(t_i) = \frac{\alpha}{1-\alpha}K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i \circ \mathcal{R}_i(t_i))v_i \circ \mathcal{R}_i(t_i), \tag{2.21}$$

for $t_i \in [s_i, 1], i = 1,2$.

In order to simplify these assumptions, it is necessary to put some restrictions on α and $K_i, i = 1,2$. Let us take $\alpha = 1/2$ and investigate a class of kernels, K_i, of the form

$$K_i(u,v) = k_i(|u-v|), \quad u, v \in \phi_i(I), i = 1,2, \tag{2.22}$$

where $k_i : [0, \infty) \to [0, \infty)$ for $i = 1,2$ are given so that the integral operators (2.16)-(2.19) are well-defined.

We check at once that

$$|\phi_i \circ \mathcal{L}_i(s_i) - \phi_i \circ \mathcal{L}_i(t_i)| = |\phi_i \circ \mathcal{R}_i(s_i) - \phi_i \circ \mathcal{R}_i(t_i)| = \frac{1}{2}|s_i - t_i||\phi(b_i) - \phi(a_i)|, i = 1,2$$
$$|\phi_i \circ \mathcal{L}_i(s_i) - \phi_i \circ \mathcal{R}_i(t_i)| = |\phi_i \circ \mathcal{R}_i(s_i) - \phi_i \circ \mathcal{L}_i(t_i)| = \frac{1}{2}(s_i + t_i)|\phi(b_i) - \phi(a_i)|, i = 1,2.$$

Consequently, (2.20) and (2.21) reduce to

$$u_i \circ \mathcal{L}_i(t_i) = u_i \circ \mathcal{R}_i(t_i), \quad t_i \in [0, s_i], i = 1,2$$

and

$$v_i \circ \mathcal{L}_i(t_i) = v_i \circ \mathcal{R}_i(t_i), \quad t_i \in [s_i, 1], i = 1,2$$

respectively. This enables one to take

$$m_i(x_i) = \begin{cases} u_i(x_i) & \text{if} \quad x_i \in [\mathcal{L}_i(s_i), \mathcal{R}_i(s_i)], \\ v_i(x_i) & \text{otherwise}, \end{cases}$$

for $i = 1, 2$. Put this way, we have

$$\beta_i(s_i) = \frac{\mathcal{K}^{\phi_i}_{\mathcal{L}_i(s_i)+}[\varphi_i m_i](\mathcal{R}_i(s_i)) + \mathcal{K}^{\phi_i}_{\mathcal{R}_i(s_i)-}[\varphi_i m_i](\mathcal{L}_i(s_i))}{\mathcal{K}^{\phi_i}_{\mathcal{L}_i(s_i)+}[m_i](\mathcal{R}_i(s_i)) + \mathcal{K}^{\phi_i}_{\mathcal{R}_i(s_i)-}[m_i](\mathcal{L}_i(s_i))},$$

$$\gamma_i(s_i) = \frac{\mathcal{K}^{\phi_i}_{a+}[\varphi_i m_i](\mathcal{L}_i(s_i)) + \mathcal{K}^{\phi_i}_{b-}[\varphi_i m_i](\mathcal{R}_i(s_i))}{\mathcal{K}^{\phi_i}_{a+}[m_i](\mathcal{L}_i(s_i)) + \mathcal{K}^{\phi_i}_{b-}[m_i](\mathcal{R}_i(s_i))},$$

where

$$\varphi_i(x_i) = \left| \frac{\phi_i(a_i) + \phi_i(b_i) - 2\phi_i(x_i)}{\phi_i(b_i) - \phi_i(a_i)} \right|, \quad x_i \in I, i = 1, 2. \tag{2.23}$$

Set

$$\begin{aligned} \mathcal{K}^1(s) = \mathcal{C}^1(s) \Big(& \mathcal{K}^{\phi_1,\phi_2}_{\mathcal{L}_1(s_1)+,\mathcal{L}_2(s_2)+}[(\psi \circ f) m_1 m_2](\mathcal{R}_1(s_1), \mathcal{R}_2(s_2)) \\ & + \mathcal{K}^{\phi_1,\phi_2}_{\mathcal{L}_1(s_1)+,\mathcal{R}_2(s_2)-}[(\psi \circ f) m_1 m_2](\mathcal{R}_1(s_1), \mathcal{L}_2(s_2)) \\ & + \mathcal{K}^{\phi_1,\phi_2}_{\mathcal{R}_1(s_1)-,\mathcal{L}_2(s_2)+}[(\psi \circ f) m_1 m_2](\mathcal{L}_1(s_1), \mathcal{R}_2(s_2)) \\ & + \mathcal{K}^{\phi_1,\phi_2}_{\mathcal{R}_1(s_1)-,\mathcal{R}_2(s_2)-}[(\psi \circ f) m_1 m_2](\mathcal{L}_1(s_1), \mathcal{L}_2(s_2)) \Big) \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}^1(s) = & \frac{1}{\mathcal{K}^{\phi}_{\mathcal{L}_1(s_1)+}[m_1](\mathcal{R}_1(s_1)) + \mathcal{K}^{\phi}_{\mathcal{R}_1(s_1)-}[m_1](\mathcal{L}_1(s_1))} \\ & \times \frac{1}{\mathcal{K}^{\phi}_{\mathcal{L}_2(s_2)+}[m_2](\mathcal{R}_2(s_2)) + \mathcal{K}^{\phi}_{\mathcal{R}_2(s_2)-}[m_2](\mathcal{L}_2(s_2))} \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}^2(s) = \mathcal{C}^1(s) \Big(& \mathcal{K}^{\phi_1,\phi_2}_{a_1+,a_2+}[(\psi \circ f) m_1 m_2](\mathcal{L}_1(s_1), \mathcal{L}_2(s_2)) \\ & + \mathcal{K}^{\phi_1,\phi_2}_{a_1+,b_2-}[(\psi \circ f) m_1 m_2](\mathcal{L}_1(s_1), \mathcal{R}_2(s_2)) \\ & + \mathcal{K}^{\phi_1,\phi_2}_{b_1-,a_2+}[(\psi \circ f) m_1 m_2](\mathcal{R}_1(s_1), \mathcal{L}_2(s_2)) \\ & + \mathcal{K}^{\phi_1,\phi_2}_{b_1-,b_2-}[(\psi \circ f) m_1 m_2](\mathcal{R}_1(s_1), \mathcal{R}_2(s_2)) \Big) \end{aligned}$$

where

$$\mathcal{C}^2(s) = \frac{1}{\mathcal{K}^{\phi}_{a_1+}[m_1](\mathcal{L}_1(s_1)) + \mathcal{K}^{\phi}_{b_1-}[m_1](\mathcal{R}_1(s_1))} \times \frac{1}{\mathcal{K}^{\phi}_{a_2+}[m_2](\mathcal{L}_2(s_2)) + \mathcal{K}^{\phi}_{b_2-}[m_2](\mathcal{R}_2(s_2))}$$

In summary, we get the following corollary.

Corollary 2.9. *Let φ be given by (2.23) and $\alpha = 1/2$. Suppose that $K_i : \phi_i(I) \times \phi_i(I) \to [0, \infty)$ is of the form (2.22) and $m_i : [a_i, b_i] \to [0, \infty)$ is integrable such that*

$$\frac{\mathcal{K}^{\phi_i}_{\mathcal{L}_i(s_i)+}[m_i](\mathcal{R}_i(s_i)) + \mathcal{K}^{\phi_i}_{\mathcal{R}_i(s_i)-}[m_i](\mathcal{L}_i(s_i))}{\phi_i(b_i) - \phi_i(a_i)} > 0 \quad \text{and} \quad \frac{\mathcal{K}^{\phi_i}_{a+}[m_i](\mathcal{L}_i(s_i)) + \mathcal{K}^{\phi_i}_{b-}[m_i](\mathcal{R}_i(s_i))}{\phi_i(b_i) - \phi_i(a_i)} > 0$$

for all $s_i \in (0, 1)$. If

$$m_i \circ \mathcal{L}_i(t_i) = m_i \circ \mathcal{R}_i(t_i), \quad t_i \in [0, 1], \tag{2.24}$$

then

$$\begin{aligned} & f(\mathcal{M}_{\phi_1}(a_1, b_2), \mathcal{M}_{\phi_2}(a_2, b_2)) \\ & \le \mathcal{F}(\beta_1(s_1), \beta_2(s_2)) \\ & \le \psi^{-1}\left(\mathcal{K}^1(s)\right) \\ & \le \mathcal{G}(\beta_1(s_1), \beta_2(s_2)) \le \mathcal{F}(s) \\ & \le \mathcal{G}(s) \le \mathcal{F}(\gamma_1(s_1), \gamma_2(s_2)) \\ & \le \psi^{-1}\left(\mathcal{K}^2(s)\right) \\ & \le \mathcal{M}_\psi(f(a), f(b)). \end{aligned} \tag{2.25}$$

In particular, one has

$$\begin{aligned} & f(\mathcal{M}_{\phi_1}(a_1, b_2), \mathcal{M}_{\phi_2}(a_2, b_2)) \\ & \le \mathcal{F}\left(\frac{\mathcal{K}^{\phi_1}_{a_1+}[\varphi_1 m_1](b_1) + \mathcal{K}^{\phi_1}_{b_1-}[\varphi_1 m_1](a_1)}{\mathcal{K}^{\phi_1}_{a_1+}[m_1](b_1) + \mathcal{K}^{\phi_1}_{b_1-}[m_1](a_1)}, \frac{\mathcal{K}^{\phi_2}_{a_2+}[\varphi_2 m_2](b_2) + \mathcal{K}^{\phi_2}_{b_2-}[\varphi_2 m_2](a_2)}{\mathcal{K}^{\phi_2}_{a_2+}[m_2](b_2) + \mathcal{K}^{\phi_2}_{b_2-}[m_2](a_2)}\right) \\ & \le \psi^{-1}\Bigg(\frac{\mathcal{K}^{\phi_1,\phi_2}_{a_1+,a_2+}[(\psi \circ f) m_1 m_2](b_1, b_2) + \mathcal{K}^{\phi_1,\phi_2}_{a_1+,b_2-}[(\psi \circ f) m_1 m_2](b_1, a_2)}{\left[\mathcal{K}^{\phi_1}_{a_1+}[m_1](b_1) + \mathcal{K}^{\phi_1}_{b_1-}[m_1](a_1)\right]\left[\mathcal{K}^{\phi_2}_{a_2+}[m_2](b_2) + \mathcal{K}^{\phi_2}_{b_2-}[m_2](a_2)\right]} \\ & \qquad + \frac{\mathcal{K}^{\phi_1,\phi_2}_{b_1-,a_2+}[(\psi \circ f) m_1 m_2](a_1, b_2) + \mathcal{K}^{\phi_1,\phi_2}_{b_1-,b_2-}[(\psi \circ f) m_1 m_2](a_1, a_2)}{\left[\mathcal{K}^{\phi_1}_{a_1+}[m_1](b_1) + \mathcal{K}^{\phi_1}_{b_1-}[m_1](a_1)\right]\left[\mathcal{K}^{\phi_2}_{a_2+}[m_2](b_2) + \mathcal{K}^{\phi_2}_{b_2-}[m_2](a_2)\right]}\Bigg) \\ & \le \mathcal{G}\left(\frac{\mathcal{K}^{\phi_1}_{a_1+}[\varphi_1 m_1](b_1) + \mathcal{K}^{\phi_1}_{b_1-}[\varphi_1 m_1](a_1)}{\mathcal{K}^{\phi_1}_{a_1+}[m_1](b_1) + \mathcal{K}^{\phi_1}_{b_1-}[m_1](a_1)}, \frac{\mathcal{K}^{\phi_2}_{a_2+}[\varphi_2 m_2](b_2) + \mathcal{K}^{\phi_2}_{b_2-}[\varphi_2 m_2](a_2)}{\mathcal{K}^{\phi_2}_{a_2+}[m_2](b_2) + \mathcal{K}^{\phi_2}_{b_2-}[m_2](a_2)}\right) \\ & \le \mathcal{M}_\psi(f(a), f(b)) \end{aligned} \tag{2.26}$$

and

$$
\begin{aligned}
&f(\mathcal{M}_1,\mathcal{M}_2)\\
&\le \mathcal{F}\left(\frac{\mathcal{K}^{\phi}_{a_1+}[\varphi_1 m_1](\mathcal{M}_1)+\mathcal{K}^{\phi_1}_{b_1-}[\varphi_1 m_1](\mathcal{M}_1)}{\mathcal{K}^{\phi_1}_{a_1+}[m_1](\mathcal{M}_1)+\mathcal{K}^{\phi_1}_{b_1-}[m_1](\mathcal{M}_1)},\frac{\mathcal{K}^{\phi_2}_{a_2+}[\varphi_2 m_2](\mathcal{M}_2)+\mathcal{K}^{\phi_2}_{b_2-}[\varphi_2 m_2](\mathcal{M}_2)}{\mathcal{K}^{\phi_2}_{a_2+}[m_2](\mathcal{M}_2)+\mathcal{K}^{\phi_2}_{b_2-}[m_2](\mathcal{M}_2)}\right)\\
&\le \psi^{-1}\left(\frac{\mathcal{K}^{\phi_1,\phi_2}_{a_1+,a_2+}[(\psi\circ f)m_1m_2](\mathcal{M}_1,\mathcal{M}_2)+\mathcal{K}^{\phi_1,\phi_2}_{a_1+,b_2-}[(\psi\circ f)m_1m_2](\mathcal{M}_1,\mathcal{M}_2)}{\left[\mathcal{K}^{\phi_1}_{a_1+}[m_1](\mathcal{M}_1)+\mathcal{K}^{\phi_1}_{b_1-}[m_1](\mathcal{M}_1)\right]\left[\mathcal{K}^{\phi_2}_{a_2+}[m_2](\mathcal{M}_2)+\mathcal{K}^{\phi_2}_{b_2-}[m_2](\mathcal{M}_2)\right]}\right.\\
&\qquad\left.+\frac{\mathcal{K}^{\phi_1,\phi_2}_{b_1-,a_2+}[(\psi\circ f)m_1m_2](\mathcal{M}_1,\mathcal{M}_2)+\mathcal{K}^{\phi_1,\phi_2}_{b_1-,b_2-}[(\psi\circ f)m_1m_2](\mathcal{M}_1,\mathcal{M}_2)}{\left[\mathcal{K}^{\phi_1}_{a_1+}[m_1](\mathcal{M}_1)+\mathcal{K}^{\phi_1}_{b_1-}[m_1](\mathcal{M}_1)\right]\left[\mathcal{K}^{\phi_2}_{a_2+}[m_2](\mathcal{M}_2)+\mathcal{K}^{\phi_2}_{b_2-}[m_2](\mathcal{M}_2)\right]}\right)\\
&\le \mathcal{G}\left(\frac{\mathcal{K}^{\phi}_{a_1+}[\varphi_1 m_1](\mathcal{M}_1)+\mathcal{K}^{\phi_1}_{b_1-}[\varphi_1 m_1](\mathcal{M}_1)}{\mathcal{K}^{\phi_1}_{a_1+}[m_1](\mathcal{M}_1)+\mathcal{K}^{\phi_1}_{b_1-}[m_1](\mathcal{M}_1)},\frac{\mathcal{K}^{\phi_2}_{a_2+}[\varphi_2 m_2](\mathcal{M}_2)+\mathcal{K}^{\phi_2}_{b_2-}[\varphi_2 m_2](\mathcal{M}_2)}{\mathcal{K}^{\phi_2}_{a_2+}[m_2](\mathcal{M}_2)+\mathcal{K}^{\phi_2}_{b_2-}[m_2](\mathcal{M}_2)}\right)\\
&\le \mathcal{M}_\psi(f(a),f(b)).
\end{aligned}
\tag{2.27}
$$

where $\mathcal{M}_1=\mathcal{M}_{\phi_1}(a_1,b_1)$ *and* $\mathcal{M}_2=\mathcal{M}_{\phi_2}(a_2,b_2)$.

Remark 2.10. Through a proper choice of the functions ϕ, ψ and K such as are indicated in Remark 2.8, (2.26) can be regarded as a generalization and refinement of several results obtained recently by Chen [6], Sarıkaya [41].

3. An application to inequalities involving the beta function

We devote this section to establish some inequalities involving the beta function, defined by the integral representation

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\quad x>0,y>0.$$

It is well-known that the beta function is log-convex on $(0,\infty)^2$ as a function of two variables (see [11]). So it is co-ordinates log-convex on $(0,\infty)^2$.

Fix $a>0$. Applying Theorem 2.1 for $f(x,y)=\beta(x,y)$, $\phi(x)=x$, $\psi(x)=\ln x$, $a_1=a_2=a, b_1=b_2=a+1$ and $\alpha=1/2$ we obtain the following result.

Corollary 3.1. 1. *The functions*

$$
\begin{aligned}
\mathcal{F}_3(t_1,t_2)=&\sqrt[4]{\beta\left(a+\frac{1-t_1}{2},a+\frac{1-t_2}{2}\right)\beta\left(a+\frac{1-t_1}{2},a+\frac{1+t_2}{2}\right)}\\
&\times\sqrt[4]{\beta\left(a+\frac{1+t_1}{2},a+\frac{1-t_2}{2}\right)\beta\left(a+\frac{1+t_1}{2},a+\frac{1+t_2}{2}\right)}
\end{aligned}
$$

and

$$\mathcal{G}_3(t_1,t_2) = \left[\beta(a,a)\sqrt[4]{\frac{a^2}{8a(2a+1)}}\right]^{t_1t_2}\left[\beta\left(a+\frac{1}{2},a\right)\sqrt{\frac{a}{2a+\frac{1}{2}}}\right]^{t_1(1-t_2)+(1-t_1)t_2}$$
$$\times\left[\beta\left(a+\frac{1}{2},a+\frac{1}{2}\right)\right]^{(1-t_1)(1-t_2)}$$

are co-ordinates log-convex and co-ordinates increasing on $[0,1]$*, with*

$$\beta\left(a+\frac{1}{2},a+\frac{1}{2}\right) \le \beta\left(a+\frac{1}{2},a\right)\sqrt{\frac{a}{2a+\frac{1}{2}}} \le \mathcal{F}_3(t_1,t_2) \le \mathcal{G}_3(t_1,t_2) \le \beta(a,a)\sqrt[4]{\frac{a^2}{8a(2a+1)}}$$

for all $t_1,t_2 \in [0,1]$.

2. *The function*

$$\mathcal{P}_3(s_1,s_2) = \exp\left(\frac{1}{s_1s_2}\int_{a+(1-s_1)/2}^{a+(1+s_1)/2}\int_{a+(1-s_2)/2}^{a+(1+s_2)/2}\ln\beta(x,y)dxdy\right)$$

is co-ordinates increasing on $(0,1]^2$*, with*

$$\lim_{s_1\to0^+}\lim_{s_2\to0^+}\mathcal{P}_3(s_1,s_2) = \beta\left(a+\frac{1}{2},a+\frac{1}{2}\right), \quad \mathcal{P}_3(1,1) = 2\pi\left(\frac{a}{e}\right)^{2a}\sqrt{\frac{(2a)^{(2a)^2}}{(2a+1)^{(2a+1)^2}}}e^{a+\frac{1}{4}},$$

and

$$\mathcal{F}_3(s_1/2,s_2/2) \le \mathcal{P}_3(s_1,s_2) \le \mathcal{G}_3(s_1/2,s_2/2) \le \mathcal{F}_3(s_1,s_2), \quad s_1,s_2 \in (0,1].$$

3. *The function*

$$\mathcal{Q}_3(s_1,s_2) = \exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_a^{a+(1-s_1)/2}\int_a^{a-(1-s_2)/2}\ln\beta(x,y)dxdy\right)$$
$$\times\exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_a^{a+(1-s_1)/2}\int_{a+(1-s_2)/2}^{a+1}\ln\beta(x,y)dxdy\right)$$
$$\times\exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_{a+(1+s_1)/2}^{a+1}\int_a^{a-(1-s_2)/2}\ln\beta(x,y)dxdy\right)$$
$$\times\exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_{a+(1+s_1)/2}^{a+1}\int_{a+(1-s_2)/2}^{a+1}\ln\beta(x,y)dxdy\right)$$

is co-ordinates increasing on $[0,1)^2$*, with*

$$\mathcal{Q}_3(0,0) = 2\pi\left(\frac{a}{e}\right)^{2a}\sqrt{\frac{(2a)^{(2a)^2}}{(2a+1)^{(2a+1)^2}}}e^{a+\frac{1}{4}}, \quad \lim_{s_1\to1^-}\lim_{s_2\to1^-}\mathcal{Q}_3(s_1,s_2) = \beta(a,a)\sqrt[4]{\frac{a^2}{8a(2a+1)}}$$

and

$$\mathcal{G}_3(s_1,s_2) \le \mathcal{F}_3\left(\frac{1+s_1}{2},\frac{1+s_2}{2}\right) \le \mathcal{Q}_3(s_1,s_2) \le \mathcal{G}_3\left(\frac{1+s_1}{2},\frac{1+s_2}{2}\right), \quad s_1,s_2\in[0,1).$$

4. *In particular,*

$$\beta\left(a+\frac{1}{2},a+\frac{1}{2}\right) \le \mathcal{F}_3(1/2,1/2)$$
$$\le 2\pi\left(\frac{a}{e}\right)^{2a}\sqrt{\frac{(2a)^{(2a)^2}}{(2a+1)^{(2a+1)^2}}}e^{a+\frac{1}{4}} \le \mathcal{G}_3(1/2,1/2) \le \beta(a,a)\sqrt[4]{\frac{a^2}{8a(2a+1)}}.$$

Next, we consider the Mittag-Leffler function, defined by the integral representation

$$E_z(x,y)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(xn+y)}, \quad z\in\mathbb{C}, x,y>0.$$

The function $\mathbb{E}(x,y)=\Gamma(x)\Gamma(y)E_z(x,y)$ (see Mehrez and Sitnik [18]) is co-ordinates log-convex on $(0,+\infty)\times(0,+\infty)$.

Fix $a>0$. Applying Theorem 2.1 for $f(x,y)=\mathbb{E}(x,y)$, $\phi(x)=x$, $\psi(x)=\ln x$, $a_1=a_2=a, b_1=b_2=a+1$ and $\alpha=1/2$ we obtain the following result.

Corollary 3.2. 1. *The functions*

$$\mathcal{F}_3(t_1,t_2)=\sqrt[4]{\mathbb{E}\left(a+\frac{1-t_1}{2},a+\frac{1-t_2}{2}\right)\mathbb{E}\left(a+\frac{1-t_1}{2},a+\frac{1+t_2}{2}\right)}$$
$$\times\sqrt[4]{\mathbb{E}\left(a+\frac{1+t_1}{2},a+\frac{1-t_2}{2}\right)\mathbb{E}\left(a+\frac{1+t_1}{2},a+\frac{1+t_2}{2}\right)}$$

and

$$\mathcal{G}_3(t_1,t_2)=\left[a\Gamma^2(a)\sqrt[4]{E_z(a,a)E_z(a,a+1)E_z(a+1,a)E_z(a+1,a+1)}\right]^{t_1t_2}$$
$$\times\left[\sqrt{a}\Gamma\left(a+\frac{1}{2}\right)\Gamma(a)\sqrt{E_z\left(a+\frac{1}{2},a\right)E_z\left(a+\frac{1}{2},a+1\right)}\right]^{t_1(1-t_2)}$$
$$\times\left[\sqrt{a}\Gamma\left(a+\frac{1}{2}\right)\Gamma(a)\sqrt{E_z\left(a,a+\frac{1}{2}\right)E_z\left(a+1,a+\frac{1}{2}\right)}\right]^{(1-t_1)t_2}$$
$$\times\left[\Gamma\left(a+\frac{1}{2}\right)E_z\left(a+\frac{1}{2},a+\frac{1}{2}\right)\right]^{(1-t_1)(1-t_2)}$$

are co-ordinates log-convex and co-ordinates increasing on $[0,1]$*, with*

$$\begin{aligned}
&\Gamma\left(a+\frac{1}{2}\right)E_z\left(a+\frac{1}{2},a+\frac{1}{2}\right)\\
&\leq \sqrt{a}\Gamma\left(a+\frac{1}{2}\right)\Gamma(a)\sqrt{E_z\left(a,a+\frac{1}{2}\right)E_z\left(a+1,a+\frac{1}{2}\right)}\\
&\leq \mathcal{F}_3(t_1,t_2)\leq \mathcal{G}_3(t_1,t_2)\\
&\leq a\Gamma^2(a)\sqrt[4]{E_z(a,a)E_z(a,a+1)E_z(a+1,a)E_z(a+1,a+1)}
\end{aligned}$$

for all $t_1,t_2\in[0,1]$*.*

2. *The function*

$$\mathcal{P}_3(s_1,s_2)=\exp\left(\frac{1}{s_1s_2}\int_{a+(1-s_1)/2}^{a+(1+s_1)/2}\int_{a+(1-s_2)/2}^{a+(1+s_2)/2}\ln\mathbb{E}(x,y)dxdy\right)$$

is co-ordinates increasing on $(0,1]^2$*, with*

$$\lim_{s_1\to0^+}\lim_{s_2\to0^+}\mathcal{P}_3(s_1,s_2)=\Gamma\left(a+\frac{1}{2}\right)E_z\left(a+\frac{1}{2},a+\frac{1}{2}\right),$$

$$\mathcal{P}_3(1,1)=2\sqrt{2\pi}\left(\frac{a}{e}\right)^a\exp\left(\int_a^{a+1}\int_a^{a+1}\ln E_z(x,y)dxdy\right),$$

and

$$\mathcal{F}_3(s_1/2,s_2/2)\leq\mathcal{P}_3(s_1,s_2)\leq\mathcal{G}_3(s_1/2,s_2/2)\leq\mathcal{F}_3(s_1,s_2),\quad s_1,s_2\in(0,1].$$

3. *The function*

$$\begin{aligned}
\mathcal{Q}_3(s_1,s_2)=&\exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_a^{a+(1-s_1)/2}\int_a^{a+(1-s_2)/2}\ln\mathbb{E}(x,y)dxdy\right)\\
&\times\exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_a^{a+(1-s_1)/2}\int_{a+(1-s_2)/2}^{a+1}\ln\mathbb{E}(x,y)dxdy\right)\\
&\times\exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_{a+(1+s_1)/2}^{a+1}\int_a^{a+(1-s_2)/2}\ln\mathbb{E}(x,y)dxdy\right)\\
&\times\exp\left(\frac{1}{(1-s_1)(1-s_2)}\int_{a+(1+s_1)/2}^{a+1}\int_{a+(1-s_2)/2}^{a+1}\ln\mathbb{E}(x,y)dxdy\right)
\end{aligned}$$

is co-ordinates increasing on $[0,1)^2$*, with*

$$\mathcal{Q}_3(0,0)=2\sqrt{2\pi}\left(\frac{a}{e}\right)^a\exp\left(\int_a^{a+1}\int_a^{a+1}\ln E_z(x,y)dxdy\right),$$

$$\lim_{s_1\to 1^-}\lim_{s_2\to 1^-}\mathcal{Q}_3(s_1,s_2)=a\Gamma^2(a)\sqrt[4]{E_z(a,a)E_z(a,a+1)E_z(a+1,a)E_z(a+1,a+1)}$$

and

$$\mathcal{G}_3(s_1,s_2)\le\mathcal{F}_3\left(\frac{1+s_1}{2},\frac{1+s_2}{2}\right)\le\mathcal{Q}_3(s_1,s_2)\le\mathcal{G}_3\left(\frac{1+s_1}{2},\frac{1+s_2}{2}\right),\quad s_1,s_2\in[0,1).$$

4. *In particular,*

$$\begin{aligned}\Gamma\left(a+\frac{1}{2}\right)E_z\left(a+\frac{1}{2},a+\frac{1}{2}\right)&\le\mathcal{F}_3(1/2,1/2)\\ &\le 2\sqrt{2\pi}\left(\frac{a}{e}\right)^a\exp\left(\int_a^{a+1}\int_a^{a+1}\ln E_z(x,y)dxdy\right)\\ &\le\mathcal{G}_3(1/2,1/2)\\ &\le a\Gamma^2(a)\sqrt[4]{E_z(a,a)E_z(a,a+1)E_z(a+1,a)E_z(a+1,a+1)}.\end{aligned}$$

Similar considerations may apply to other special functions provided that these functions are co-ordinates log-convex.

(i) The Struve function, defined by the integral representation

$$M(x,y)=\frac{2}{\sqrt{\pi}}\int_0^1(1-t^2)^{x-\frac{1}{2}}e^{-yt}dt,\quad x>-\frac{1}{2},y>0.$$

is co-ordinates log-convex on $(-\frac{1}{2},+\infty)\times(0,+\infty)$ (see [26]).

(ii) The Gauss function

$$G(x,y)={}_2F_1(x,y;c;z)=\sum_{n=0}^{\infty}\frac{(x)_n(y)_n}{(c_n)n!}z^n,\quad c>x>0,c>y>0,z<1.$$

is co-ordinates log-convex on $(0,+\infty)\times(0,+\infty)$ (see [27]).

Remark 3.3. One may develop further inequalities related to the Struve function and Gauss function by applying Theorem 2.1 for $f(x,y)=M(x,y)$ and $f(x,y)=G(x,y)$, $\phi(x)=x$, $\psi(x)=\ln x$.

Acknowledgement

The first author would like to thank the Vietnam Institute for Advanced Study in Mathematics for support during the visit at VIASM.

References

[1] J. Aczél, The notion of mean values, Norske Vid. Selsk. Forhdl. 19 (1947) 83–86.
[2] G. Aumann, Konvexe Funktionen und die Induktion bei Ungleichungen zwischen Mittelwerten, Sitzungsber., Bayer. Akad. Wiss., Math.-Naturwiss. Kl. 1933 (1933) 403–415.

[3] M.K. Bakula, An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates, Aust. J. Math. Anal. Appl. 11(1) (2014), Art. 3.
[4] J.L. Brenner, H. Alzer, Integral inequalities for concave functions with applications to special functions, Proc. Roy. Soc. Edinburgh Sect. A 118 (1991) 173–192.
[5] P.S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers Group, Dordrecht, 2003.
[6] F. Chen, On Hermite-Hadamard type inequalities for s-convex functions on the coordinates via Riemann-Liouville fractional integrals, J. Appl. Math. Volume 2014, Article ID 248710, 8 pages
[7] F. Chen, S. Wu, Fejér and Hermite-Hadamard type inequalities for harmonically convex functions, J. Appl. Math. 2014 (2014) 386806.
[8] D.T. Duc, N.N. Hue, N.D.V. Nhan, V.T. Tuan, Convexity according to a pair of quasi-arithmetic means and inequalities, J. Math. Anal. Appl. 488 (2020) 124059.
[9] S.S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl. 167 (1992) 49–56.
[10] S.S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese J. Math., 5 (2001) 775–788.
[11] S.S. Dragomir, R.P. Agarwal, N.S. Barnett, Inequalities for beta and gamma functions via some classical and new integral inequalities. J. Inequal. Appl. 5 (2000) 103–165.
[12] S.S. Dragomir, B. Mond, Integral inequalities of Hadamard's type for log-convex functions, Demonstr. Math. 31 (1998) 354–364.
[13] S.S. Dragomir, D.S. Milošević, J. Sándor, On some refinements of Hadamard's inequalities and applications, Univ. Belgrad Publ. Elek. Fak. Sci. Math. 4 (1993) 3–10.
[14] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2002.
[15] G. Farid, M. Marwan, Atiq Ur Rehman, Fejér-Hadamard Inequlality for Convex Functions on the Coordinates in a Rectangle from the Plane, International Journal of Analysis and Applications, Volume 10, Number 1 (2016), 40-47
[16] L. Fejér, Über die Fourierreihen, II, Math. Naturwiss. Anz. Ungar. Akad. Wiss. 24 (1906) 369–390.
[17] P.M. Gill, C.E.M. Pearce, J. Pečarić, Hadamard's inequality for r-convex functions, J. Math. Anal. Appl. 215 (1997) 461–470.
[18] K. Mehrez and S.M. Sitnik, Functional Inequalities for the Mittag-Leffler Functions, Results in Mathematics. 72 (2017) 703–714.
[19] J. Hadamard, Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann, J. Math. Pures Appl. 58 (1893) 171–215.
[20] P.C. Hammer, The midpoint method of numerical integration, Math. Mag. 31 (1958) 193–195.
[21] Ch. Hermite, Sur deux limites d'une intégrale dé finie, Mathesis 3 (1883), 82.
[22] D.-Y. Hwang, K.-L. Tseng, G.-S. Yang, Some Hadamard's inequalities for co-ordinated convex functions in the rectangle from the plane, Taiwanese J. Math. 11 (2007) 63–73.
[23] K.Ch. Hsu Refinements of Hermite-Hadamard inequalities for differenttiable co-ordinated convex functions and applications, Taiwanese J. Math. 19 (2015) 133– 157.
[24] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacet. J. Math. Stat. 43 (2014) 935–942.
[25] İ. İşcan, Hermite-Hadamard type inequalities for p-convex functions, Int. J. Anal. Appl. 11 (2016) 137–145.
[26] A. Baricz and T.K. Pogány, Functional inequalities for modified Struve functions II, Proceedings of the Royal Society of Edinburgh Section A Mathematics 144(5) (2014) 891-904
[27] D. Karp and S.M. Sitnik, Log-convexity and log-concavity of hypergeometric-like functions, J. Math. Anal. Appl. 364 (2010) 384–394
[28] D. Kotrys, Hermite-Hadamard inequality for convex stochatic processes, Aeqquat. Math. 83 (2012) 143–151.
[29] M. Kunt, İ. İşcan, Hermite-Hadamard-Fejér type inequalities for p-convex functions, Arab J. Math.

Sci. 23 (2017) 215–230.
[30] A. Lupaş, A generalisation of Hadamard's inequalities for convex functions, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 544–576 (1976) 115–121.
[31] M. Merkle, Logarithmic convexity and inequalities for the gamma function, J. Math. Anal. Appl. 203 (1996) 369–380.
[32] K. Nikodem, On convex stochatic processes, Aequat. Math. 20 (1980) 184–197.
[33] C.P. Niculescu, L.-E. Persson, Convex Functions and Their Applications: A Contemporary Approach, second ed., Springer International Publishing AG, part of Springer Nature, Switzerland, 2018.
[34] M.A. Noor, K.I. Noor, M.U. Awan, Some characterizations of harmonically log-convex functions, Proc. Jangjeon Math. Soc. 17 (2014) 51–61.
[35] M.A. Noor, K.I. Noor, M.U. Awan, Some new estimates of Hermite-Hadamard inequalities via harmonically r-convex functions, Matematiche 71 (2016) 117–127.
[36] Eze R. Nwaeze, Generalized Hermite-Hadamard inequality for functions convex on the coordinates, Applied Mathematics E- Notes, 17 (2017) 117–127.
[37] M.E. Özdemir, C. Yildiz, A.O. Akdemir, On some new the Hadamard-type inequalities for co-ordinated quasi-convex functions, Hacet. J. Math. Stat. 41(5), 697–707 (2012)
[38] J.E. Pečarić, F. Proschan, Y.C. Tong, Convex Functions, Partial Orderings and Statistical Applications, Mathematics in Science and Engineering 187, Academic Press, Boston, 1992.
[39] J. Park, Generalized inequalities for convex mappings on the co-ordinates, Inter. J.of Pure and Applied Math. (IJPAM) 82 (2013) 829–842.
[40] J.L. Raabe, Angenäherte Bestimmung der Factorenfolge $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 ... n = \Gamma(n+1) = \int x^n e^{-x} dx$, wenn n eine sehr grosse Zahl ist, J. Reine Angew. Math. 25 (1840) 146–159.
[41] M.Z. Sarıkaya, On the Hermite–Hadamard-type inequalities for co-ordinated convex function via fractional integrals, Integral Transforms and Special Functions, 2014 25(2) 134–147,
[42] E. Set, Hermite-Hadamard type inequalities for coodinates convex stochatics processes, Mathematica Aeternal. 5 (2015) 363–382.
[43] T. Trif, Characterizations of convex functions of a vector variable via Hermite-Hadamard's inequality, J. Math. Inequal. 2 (2008) 37–44.
[44] T. Trif, Convexity of the gamma function with respect to Hölder means, in: Y.J. Cho, J.K. Kim, S.S. Dragomir (Eds.), Inequality Theory and Applications 3, Nova Science Publishers, New York, 2003, pp. 189–195.
[45] K.-L. Tseng, S.-R. Hwang, S.S. Dragomir, Fejér-type inequalities (I), J. Inequal. Appl. 2010 (2010) 531976.
[46] K.-L. Tseng, S.-R. Hwang, S.S. Dragomir, Fejér-type inequalities (II), Math. Slovaca 67 (2017) 109–120.
[47] P.M. Vasić, I.B. Lacković, On an inequality for convex functions, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 461–497 (1974) 63–66.
[48] P.M. Vasić, I.B. Lacković, Some complements to the paper "On an inequality for convex functions", Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 544–576 (1976) 59–62.
[49] G.-S. Yang, M.X. Hong, A note on Hadamard's inequality, Tamkang J. Math. 28 (1997) 33–37.
[50] G.-S. Yang, K.-L. Tseng, On certain integral inequalities related to Hermite-Hadamard inequalities, J. Math. Anal. Appl. 239 (1999) 180–187.