LOWER BOUNDS ON THE BERGMAN METRIC NEAR POINTS OF INFINITE TYPE

DAU THE PHIET AND NINH VAN THU

Abstract. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ satisfying an f-property for some function f, we show that the Bergman metric associated to Ω has the lower bound $\tilde{g}(\delta_\Omega(z)^{-1})$ where $\delta_\Omega(z)$ is the distance from z to $\partial\Omega$ and $\tilde{g}$ is a specific function defined by f. This refines Khanh-Zampieri's work in [KZ12] with reducing the smoothness assumption of the boundary.

Contents

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ with the boundary $\partial\Omega$ and $\delta_\Omega(z)$ denote the distance from z to $\partial\Omega$. The Bergman metric associated to Ω at the point $z \in \Omega$ acting the vector $X \in T^{1,0}(\mathbb{C}^n)$ is defined by

$$B_\Omega(z, X) := \left(\sum_{j,k=1}^{n} \frac{\partial^2 \log K_\Omega(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \right)^{1/2}.$$

It is an interesting question is to consider how rate of the Bergman metric tends to infinity uniformly at the boundary points. When $\partial\Omega$ is C^2-smooth and Ω is either strongly pseudoconvex or pseudoconvex of finite type in $\mathbb{C}^2$, the Bergman metric $B_\Omega(z, X)$ is asymptotically equivalent to $\delta_\Omega^{-1/m}(z)|X^\tau| + \delta_\Omega^{-1}(z)|X^\nu|$ (see [Cat89, Mc92, Die70]) where X^τ and X^ν are the tangential and normal components of X and m is the type of the boundary ($m = 2$ if Ω is strongly pseudoconvex). For a general pseudoconvex domain in $\mathbb{C}^n$ with C^∞-smooth boundary, using the subelliptic estimate for the $\bar{\partial}$-Neumann problem, McNeal [Mc92] gave a lower bound of this metric with rate $\delta_\Omega^{-\epsilon}(z)$ for some $\epsilon > 0$. This result was also obtained by Chen [Che02] by using the properties of plurisubharmonic peak functions in Hölder space. Recently, Herbort [Her14] proved that if an (t^ϵ-$\tilde{P}$)-property (see below) holds for Ω then $B_\Omega(z, X)$ has the lower bound $\delta_\Omega^{-\epsilon}(z)|\log(\delta_\Omega(z))|^{-M}|X|$ for some M. The novelty of the proofs by Chen and Herbort is that no smoothness assumptions of the boundary are made.

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It should be noted that by an amalgamation of results in [Cat83, Cat87, KZ10, KZ12, Kha14], if the boundary is smooth then the finite type condition, the subelliptic estimate for the $\bar{\partial}$-Neumann problem, the existence of a family of plurisubharmonic peak functions in a Hölder space, the (t^ϵ-$\tilde{P}$)-property, and the (t^ϵ-P)-property (see below) are equivalent.

For a general pseudoconvex domain Ω in $\mathbb{C}^n$ that is not necessary of finite type nevertheless smooth boundary, Khanh-Zampieri [KZ10, KZ12] proved that if an (f-P)-property with $\frac{f(t)}{\ln t} \to +\infty$ holds for Ω then the Bergman metric has a lower bound with the rate $\frac{f}{\log}(\delta_\Omega^{-1+\eta}(z))$ for any $\eta > 0$. The aim of this paper is to improve this result by reducing the assumption of smoothness of the boundary. Here is the main result of this paper.

Theorem 1.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary $b\Omega$ and z_0 be a boundary point. Assume that Ω has an (f-P)-property at z_0 with f satisfying $\displaystyle\int_t^\infty \frac{da}{af(a)} < \infty$ for some $t \geq 1$ and denote the integral by $(g(t))^{-1}$. Then there exist a neighborhood U of z_0 and a constant $C > 0$ such that*

$$B_\Omega(z, X) \geq C.\tilde{g}(\delta_\Omega^{-1}(z))|X| \tag{1.1}$$

for any $z \in V \cap \Omega$ and $X \in T_z^{1,0}\mathbb{C}^n$, where the function $\tilde{g}$ is given by

$$\tilde{g}(t) = \sqrt[4]{g\left(t^\gamma\right)} \text{ for all } t \geq 1$$

for some constant $\gamma > 0$.

The use of plurisubharmonic peak functions also enables to weaken the smoothness assumption on the boundary.

In what follows, $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant. Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$. In addition, the superscript * denotes the inverse function.

2. The (f-P)-property and plurisubharmonic peak functions

We start this section by the definition of the (f-P)-property.

Definition 2.1. For a smooth, monotonic, increasing function $f : [1, +\infty) \to [1, +\infty)$ with $f(t)t^{-1/2}$ decreasing, we say that Ω has the (f-P)-property (or f-property for short) if there exist a neigborhood U of $b\Omega$ and a family of functions $\{\phi_\delta\}$ such that

(i) the functions ϕ_δ are plurisubharmonic, C^2 on U, and satisfy $-1 \leq \phi_\delta \leq 0$, and
(ii) $i\partial\bar{\partial}\phi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\phi_\delta| \lesssim \delta^{-1}$ for any $z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\}$, where r is a C^2-defining function of Ω.

If the bounded condition of ϕ_δ in (i) is replaced by the self-bounded gradient condition, i.e, $i\partial\bar{\partial}\phi_\delta \gtrsim i\partial\phi_\delta \otimes \bar{\partial}\phi_\delta$, we say that Ω has the (f-$\tilde{P}$)-property.

It is proven in [Cat87, Mc92] that if $\Omega \subset \mathbb{C}^n$ is of finite type, then Ω satisfies the t^ϵ-property. Therefore, the estimate (1.1) holds for $\tilde{g}(t) = t^\delta$ for some $\delta > 0$. Moreover, the f-property holds for a large class of infinite type pseudoconvex domains in $\mathbb{C}^n$, such as the following example:

Let $\Omega \subset \mathbb{C}^n$ be a domain defined by

$$\Omega = \left\{ z \in \mathbb{C}^n \colon \mathrm{Re}(z_n) + \sum_{j=1}^{n-1} P_j(z_j) < 0 \right\}, \tag{2.1}$$

where $\Delta P_j(z_j) \gtrsim \frac{\exp(-1/|x_j|^\alpha)}{x_j^2}$ or $\frac{\exp(-1/|y_j|^\alpha)}{y_j^2}$ for $j = 1, \dots, n-1$. Then the f-property holds with $f(t) = \log^{1/\alpha} t$ (see [Kha13]). As in [Kha13], we obtain the following corollary.

Corollary 2.2. a) *Let Ω be a pseudoconvex domain of finite type in $\mathbb{C}^n$. Then (1.1) holds for $\tilde{g}(t) = t^\epsilon$.*

b) *Let Ω be defined by (2.1) with $\alpha < 1$. Then (1.1) holds for $\tilde{g}(t) = \log^{\left(\frac{1}{\alpha}-1\right)/4} t$.*

The proof of Theorem 1.1 is based on the following result about the existence of a family of plurisubhamonic peak functions which was recently proven by Khanh [Kha13].

Theorem 2.3. *Under the assumptions of Theorem 1.1, for any $\zeta \in b\Omega$, there exists a C^2 plurisuhharmonic function ψ_ζ on Ω which is continuous on $\overline{\Omega}$ and peaks at ζ (that means, $\psi_\zeta(z) < 0$ for all $z \in \overline{\Omega} \setminus \{\zeta\}$ and $\psi_\zeta(\zeta) = 0$). Moreover, there are some positive constants c_1 and c_2 such that the following hold for any constant $0 < \eta < 1$:*

(1) $|\psi_\zeta(z) - \psi_\zeta(z')| \le c_1|z - z'|^\eta$ for any $z, z' \in \overline{\Omega}$; and

(2) $g\big((-\psi_\zeta(z))^{-1/\eta}\big) \le c_2|z - \zeta|^{-1}$ for any $z \in \overline{\Omega} \setminus \{\zeta\}$.

The function ψ_ζ above is called a plurisubharmonic peak function at the boundary point ζ. The following lemma follows immediately from Theorem 2.3.

Corollary 2.4. *Under the assumptions of Theorem 2.3, for any $\zeta \in b\Omega$ there are some positive constants c_1 and c_1 such that the following hold for any constant $0 < \eta < 1$:*

$$-c_1|z - \zeta|^\eta \le \psi_\zeta(z) \le -\left(\frac{1}{g^*(c_2/|z-\zeta|)}\right)^\eta,$$

where ψ_ζ is the plurisuhharmonic peak function given in Theorem 2.3.

We also need a version of L^2-estimate for the $\bar{\partial}-$equation that is generalized by Berndtsson, due to Donnelly - Fefferman [DF83]

Proposition 2.5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let φ be plurisubharmonic in Ω. Let ψ be plurisubharmonic and assume that ψ has a self-bounded gradient . Let $0 < \nu < 1$. Then for any $\bar{\partial}$-closed (0,1)-form g in Ω, there is a solution u to the equation $\bar{\partial} u = g$ such that*

$$\int_\Omega |u|^2 e^{-\varphi+\nu\psi} dV \le \frac{4}{\nu(1-\nu)^2} \int_\Omega |g|^2_{\partial\bar{\partial}\psi} e^{-\varphi+\nu\psi} dV.$$

3. Boundary behavior of the Bergman metric

Combining with the result in Theorem 2.3, we can now rephrase Theorem 1.1 in a more general setting. More precisely, we have the following theorem, which generalizes [Che02, Theorem 2].

Theorem 3.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, ζ be a given boundary point, F_1 and F_2 are positive increasing functions satisfying that the function $-\log \circ F_1^*(-t)$ is convex on $(-1, 0)$. Assume that on a neighborhood U of ζ, there is a plurisubharmonic function ρ_ζ peaking at ζ and satisfying*

$$-F_1(|z - \zeta|) \le \rho_\zeta(z) \le -F_2(|z - \zeta|) \tag{3.1}$$

for $z\in U\cap\Omega$. Then there exists $k_0>0$ such that

$$B_\Omega(z,X)\gtrsim\left(F_2^*\left(F_1((3\delta(z))^{1/k_0})\right)\right)^{-1/4}|X|.$$

Furthermore, in the case where the plurisubharmonic peak function satisfies a lower bound, we obtain the following theorem, which is a generalization of [Che02, Theorem 1].

Theorem 3.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, w_0 be a given boundary point, F is positive increasing function satisfying that the function $-\log\circ F^*(-t)$ is convex on $(-1,0)$. Assume that on a neighborhood U of w_0, there is a plurisubharmonic function ρ peaking at w_0 and satisfying*

$$-F(|z-w_0|)\leq\rho(w) \tag{3.2}$$

for $z\in U\cap\Omega$. Then

$$\inf_{0\neq X\in T^{1,0}\mathbb{C}^n}B_\Omega(z,X)/|X|\to+\infty$$

as $z\to w_0$.

The proofs of Theorem 3.1 and Theorem 3.2 are adapted from the argument by Chen [Che02] with precise the rate of lower bounds.

Let z be a fixed point in Ω, and $\pi(z)$ be the projection of z to the boundary $b\Omega$. Put $\phi(z):=-\log(F^*(-\rho(z)))$. Notice that since $-\log\circ F^*(-t)$ is convex on and increasing on $(-1,0)$, we have that ϕ is plurisubharmonic on U. Moreover, by shrinking the neighborhood U, we can assume that $F_1(|z-w|)<1$ and $F(|z-w|)<1$ for any $z\in U\cap\Omega$.

Denote by $\delta(z):=\delta_\Omega(z)=|z-\pi(z)|$ the Eucildian distance from z to $b\Omega$. For $k\in\mathbb{N}^*$, we define on Ω a function as follows

$$g_{k,w}(z)=\chi\left(\frac{1}{\log k}(-\log\phi(z)+\log(-\log\epsilon)+1)\right)\log|z-w|,$$

where χ is a $C^\infty(\mathbb{R})$ cut-off function satisfying

$$\chi(t)=\begin{cases}1 & \text{if } t\leq 0\\ 0 & \text{if } t\geq 1\end{cases}$$

such that $\sup|\chi'|\leq 2$ and $\sup|\chi''|\leq 1$.

To prove the Theorem 3.1 and Theorem 3.2, we need the following lemma.

Lemma 3.3. *Let $C>2n$ be a positive constant. Then there exists a constant $k_0>0$ depends on F and n so that for any $w\in\Omega$ with $|w-w_0|<\epsilon^{k_0}/2$ the following holds*

i) $g_{k_0,w}(w)\approx\log|z-w|$ when z near w;
ii) $4Cg_{k_0,w}(w)+\phi(z)-\log(-\log|w-z|)$ is a plurisubharmonic function on Ω.

Proof. We may assume that $|w-w_0|<\epsilon^k/2$ where k will be determined later on. Then we have

$$\rho(w)\geq-F(|w-w_0|)\geq-F(\epsilon^k/2)>-F(\epsilon^k).$$

From the definition of the cut-off function χ and $g_{k,w}$, we get

$$\left\{\zeta\in\Omega:\rho(z)>-F(\epsilon^k)\right\}\subset\left\{z\in\Omega:g_{k,w}(z)=\log|z-w|\right\}.$$

Thus we conclude that $g_{k,w}(z)\sim\log|z-w|$ near w.

A computation shows that

$$\partial\bar\partial g_{k,w} = \frac{\log|z-w|}{\phi\log k}\Big(\chi''(.)\frac{\partial\phi\bar\partial\phi}{\phi\log k} + \chi'(.)\frac{\partial\phi\bar\partial\phi}{\phi} - \chi'(.)\partial\bar\partial\phi\Big) \tag{3.3}$$

$$- \frac{\chi(.)\log|z-w|}{\phi\log k}\Big(\partial\phi\frac{\bar\partial\log|z-w|}{\log|z-w|} + \bar\partial\phi\frac{\bar{\log}|z-w|}{\log|z-w|}\Big) \tag{3.4}$$

$$+ \chi(.)\partial\bar\partial\log|z-w|. \tag{3.5}$$

It is clearly that the term (3.5) is non-negative and thus it can be neglected. For another terms, it is sufficient to consider them in the support of χ'. Moreover, we have

$$\mathrm{supp}\chi' \subset \big\{z\in\Omega : \rho(z) < -F(\epsilon^k)\big\} \subset \big\{z\in\Omega : |z-w_0| \geq \epsilon^k\big\}$$

Therefore, one obtains

$$|z-w| \geq |z-w_0| - |w-w_0| \geq \frac{1}{2}|z-w_0|$$

on $\mathrm{supp}\chi'(.)$ since $|w-w_0| < \epsilon^k/2$, and hence

$$|\phi(z)| = |\log F^*(-\rho(z))| \geq |\log|z-w_0|| \geq |\log 2|z-w||.$$

Now Cauchy-Schwarz's inequality implies that

$$\pm 2\mathrm{Re}\left(\frac{\partial\phi\bar\partial\log|z-w|}{\log|z-w|}\right) \geq -\partial\phi\bar\partial\phi - \partial\log(-\log|z-w|)\bar\partial\log(-\log|z-w|)$$

in the distribution sense.

Combining above statements, there exists a constant C' (depending only on F) so that

$$\partial\bar\partial g_{k,w}(z) \geq -\frac{C'}{\log k}\left(\partial\bar\partial\phi + \partial\bar\partial(-\log(-\log|z-w|))\right)$$

provided $\phi > \log 2$ on Ω, $\partial\bar\partial\phi \geq \partial\phi\bar\partial\phi$, and

$$\partial\bar\partial(-\log(-\log|z-w|))) \geq \partial\log(-\log|z-w|)\bar\partial\log(-\log|z-w|).$$

Therefore, if we take k_0 big enough so that $\dfrac{C'}{\log k_0} < \dfrac{1}{4C}$, then the assertion (ii) follows. □

Proof of Theorem 3.2. We shall follow the guildlines of [Che02]. First of all. we recall that

$$B_\Omega(w,X) = K^{-1/2}(w)\sup\{|Xf(w)| : f\in H^2(\Omega), f(w)=0 \text{ and } \|f\|_\Omega \leq 1\}.$$

Let $X\in T^{1,0}(\mathbb{C}^n)$. Since the Bergman metric is biholomorphically invariant, without loss of generality we may assume that $X = |X|\partial_{w_1}$.

Recall that $\phi = -\log(F^*(-\rho))$ and define

$$\delta = \delta(\epsilon) := \sup_{z\in\overline\Omega, \rho(z)\geq -F(\epsilon)} |z-w_0|.$$

Note that $\delta\to 0$ as $\epsilon\to 0$ since ρ is a plurisubharmonic peak function. Furthermore, without loss of generality we can assume that $U\cap\Omega = \Omega$, $\mathrm{diam}(\Omega)\leq e^{-1}$, and $\phi > \log 2$.

We now define

$$\eta_w = \kappa\left(-\log(-\log|z-w|) + \log(-\log\delta^{1/2}) + 1\right),$$

$$\psi_w = \frac{1}{2}\left(\phi - \log(-\log|z-w_0|)\right),$$

and

$$\varphi_w = Cg_w - \frac{1}{4}\log(-\log(|z-w|)) + \frac{1}{2}\psi_w,$$

where κ is a cut-off function such that

$$\kappa(x) = \begin{cases} 1 & \text{if } x < 1-\log 2, \\ 0 & \text{if } x > 1; \end{cases}$$

and $C > 2n$ comes from Lemma 3.3. It is easy to check by a simple compution that φ_w is plurisubharmonic, $\partial\bar\partial\psi_w \gtrsim \partial\psi_w\bar\partial\psi_w$ and $\partial\bar\partial\psi_w \gtrsim \partial\log|z-w|)\bar\partial\log|z-w|)$. Then we obtain

$$|\bar\partial\eta_w|_{\partial\bar\partial\psi_w} \leq \sup|\kappa'|.$$

Noting that $\text{supp}g_w \subset \{z\in\Omega\colon \rho(z) \geq -F(\epsilon)\} \subset \{z\in\Omega\colon \eta_w(z)=1\}$. This implies that $\eta_w(w)=1$ and $\text{supp}\bar\partial\eta_w \cap \text{supp}g_w = \emptyset$.

Now, we apply Proposition 2.5 with $\nu = \frac{1}{2}$, $\varphi=\varphi_w$ and $\psi=\psi_w$ to solve the $\bar\partial-$ equation

$$\bar\partial u_w = (z_1-w_1)\frac{K_\Omega(z,w)}{K_\Omega^{1/2}(w)}\bar\partial\eta_w,$$

on Ω with the estimate

$$\begin{aligned}
&\int_\Omega |u_w|^2 e^{-Cg_w+\frac{1}{4}\log(-\log|z-w|))}dV \\
&\qquad\lesssim \int_{\text{supp}\bar\partial\eta_w} |z_1-w_1|^2\frac{|K_\Omega^2(z,w)|}{|K_\Omega(w)|}|\bar\partial\eta_z|^2_{\partial\bar\partial\psi_w}e^{-Cg_w+\frac{1}{4}\log(-\log|z-w|))}dV \\
&\qquad\leq C_1\int_{\Omega\cap\{|z-w_0|<\delta^{1/2}\}} |z-w|^2\frac{|K_\Omega(\zeta,z)|^2}{|K_\Omega(z)|}dV \\
&\qquad\leq C_2\delta^{1/2},
\end{aligned}$$

where C_1 and C_2 are constants only depending on $\sup\kappa'$.

Since $Cg_w - \frac{1}{4}\log(-\log|z-w|) < 0$ on Ω, $g_w \sim \log|z-w|$ near w, and

$$Cg_w - \frac{1}{4}\log(-\log|z-w|) < 2n\log|z-w|$$

near w, it follows that $u_w(w)=0$ and the function

$$f_w = (z_1-w_1)\frac{K_\Omega(z,w)}{K_\Omega^{1/2}(w)}\eta_w - u_w$$

is holomorphic on Ω and satisfies

$$f_w(w) = u_w(w) = 0 \text{ and } Xf_w(z) = |X|K_\Omega^{1/2}(w);$$

$$
\begin{aligned}
\|f_w\|_\Omega &\le \left\|(z_1-w_1)\frac{K_\Omega(.,w)}{K_\Omega^{1/2}(w)}\eta_w\right\|_\Omega + \|u_w\|_\Omega \\
&\le C_3\delta^{1/2} + \|u_z\|_{1/2Cg_w - 1/2\log(-\log|z-w|)} \\
&\le C_4\delta^{1/4}.
\end{aligned}
$$

Define $h_w = \dfrac{f_w}{\|f_w\|}$. Then we have h_w is also holomorphic, $h_w(w) = 0$, and $\|h_w\| = 1$. Therefore, we conclude

$$
B_\Omega(w,X) \ge \frac{|Xh_w(w)|}{K_\Omega^{1/2}(w)} = \frac{|Xh_w(w)|}{K_\Omega^{1/2}(w)\|f_w\|_\Omega} \ge C_4^{-1}\delta^{-1/4}|X|
$$

for any $w \in \Omega \cap \{w \in \Omega \colon |w-w_0| < \epsilon^{k_0}/2\}$. So, the proof is complete. □

Proof of Theorem 3.1. We shall repeat the argument as in the proof of Theorem 3.2. For any $w \in \Omega$, let $w' := \pi(w)$. It means that $|w-w'| = \delta_\Omega(w)$. Then we take $\epsilon := (3\delta_\Omega(w))^{1/k_0}$. Hence, it is clear that $|w-w'| < \epsilon^{k_0}/2$ and

$$
\delta = \sup_{z\in\overline{\Omega}, \rho_{w'}(z) \ge -F_1(\epsilon)} |z-w'| \le F_2^*(F(\epsilon))
$$

because $-\rho_{w'}(z) \ge F_2(|z-w'|)$. Therefore, we obtain

$$
B_\Omega(w,X) \gtrsim \delta_\Omega^{-1/4}(w)|X| \gtrsim \left(F_2^*(F_1((3\delta_\Omega(w))^{1/k_0}))\right)^{-1/4}|X|,
$$

which proves the theorem. □

We now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Denote $F_1(t) := c_1 t^\eta$ and $F_2(t) := \left(\frac{1}{g^*(c_2/t)}\right)^\eta$ for all $t \ge 1$, where $0 < \eta < 1$. Then, a computation shows that

$$
F_2^*\left(F_1((3\delta(z))^{1/k_0})\right) = c_2/g\left(\frac{1}{c_1^{1/\eta}(3\delta(z))^{1/k_0}}\right).
$$

Therefore, by Corollary 2.4 and employing Theorem 3.1 for $F_1(t) = c_1t^\eta$ and $F_2(t) = \left(\frac{1}{g^*(c_2/t)}\right)^\eta$, where $\eta \in (0,1)$ is given in Corollary 2.4, we obtain

$$
B_\Omega(z,X) \gtrsim \left(g\left(\frac{1}{c_1^{1/\eta}(3\delta(z))^{1/k_0}}\right)\right)^{1/4}|X|
$$

for any $z \in V\cap\Omega$ and $X \in T_z^{1,0}\mathbb{C}^n$. Moreover, by the increasing of g and decreasing of $g(t)/t$, we conclude that

$$
B_\Omega(z,X) \gtrsim \tilde{g}(\delta_\Omega(z))|X|
$$

for any $z \in V \cap \Omega$ and $X \in T_z^{1,0}\mathbb{C}^n$, where the function $\tilde{g}$ is given by

$$
\tilde{g}(t) = \sqrt[4]{g\left(t^{1/k_0}\right)} \text{ for every } t \ge 1.
$$

Hence, the proof is complete. □

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Dau The Phiet

Faculty of Applied Science, Ho Chi Minh City University of Technology, Vietnam National University, 268 Ly Thuong Kiet, District 10, Ho Chi Minh City, Vietnam
E-mail address: dauthephiet@hcmut.edu.vn

Ninh Van Thu

Department of Mathematics, Vietnam National University at Hanoi, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam
E-mail address: thunv@vnu.edu.vn