A spine for Teichmüller space

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In this note, we will describe an equivariant spine for the Teichmüller space of a closed surface. This spine is very simply characterized as the set of hyperbolic structures on the surface for which there are enough geodesics of minimum length to fill the surface, that is, so that the complement of their union consists of contractible pieces.

Unfortunately, we do not have a combinatorial characterization of collections of curves which can be the collection of shortest geodesics on a surface. This seems like a challenging problem, and until more is understood about how to answer it, there are probably not many applications of the current result. Nonetheless, the characterization of this spine has a certain intuitive appeal, and we present it for what it is worth.

John Harer has previously described a spine for the Teichmüller space of a surface with at least one puncture. His construction has the advantage of having a purely combinatorial description, but it does not readily generalize to the case of a closed surface.

Our construction is short and simple, given the general methods of elementary differential topology.

First, we note that the condition on a hyperbolic structure that a given set of geodesics have equal length is an analytic condition, so the solution set admits a triangulation. In addition, a countable number of analytic inequalities are necessary to guarantee that the lengths of these geodesics are shorter than all other lengths. In any compact subset of Teichmüller space, however, only a finite subset of these inequalities are nontrivial, so again the set of solutions has a triangulation. The solutions for different sets of geodesics have triangulations which fit together coherently, just because of the analytic nature of the defining conditions, so that the subset P of Teichmüller space consisting of hyperbolic structures such that the set of shortest geodesics fill up the surface is a subcomplex.

Now we will construct an isotopy ϕ_t of Teichmüller space, defined for all $t > 0$, such that for any neighborhood of P and any compact set K of Teichmüller space, there is a t such that $\phi_t(K)$ is contained in the neighborhood of P. The isotopy will be respected by the mapping class group of the surface. If one prefers, ϕ_t can easily be replaced by a deformation retraction, since P is a deformation retraction of a regular neighborhood.

The only slightly original observation concerning the geometry of surfaces is the following:

0.1. Proposition: expanding subsets. *Let Γ be any collection of simple closed curves on a surface which do not fill the surface. Then there are tangent vectors to Teichmüller space which simultaneously increase the lengths of the geodesics representing curves in Γ.*

Proof, expanding subsets: Under the hypothesis, there is a decomposition of the whole surface as a union of subsurfaces with geodesic boundary, disjoint except at their boundary, such that each element of Γ is isotopic to at least one of the subsurfaces.

Consider any such surface with geodesic boundary. It has a canonical extension which is a complete hyperbolic surface. Any geodesic arc properly embedded in the surface with boundary extends to a complete embedded line in the larger surface. A new hyperbolic structure can be constructed by cutting along such a line and inserting a strip from the hyperbolic plane bounded by two nearby lines.

The new complete hyperbolic surface maps back to the old by a map which is an isometry except in the inserted strip, where it still has Lipschitz constant one but contracts to points the leaves of a foliation by equidistant curves. Clearly, any closed geodesic in the enlarged surface which intersects the strip is represented by a shorter geodesic in the original surface.

By composing this operation performed on several different arcs, every closed geodesics in the surface with boundary can be lengthened. Since arcs can be chosen to go from a boundary component back to itself, the boundary geodesics can be lengthened independently by any positive amount. Therefore, the various pieces in the decomposition of the closed surface can be modified in such a way that all their closed geodesics lengthen, and they still fit together.

The statement of the proposition is the infinitesimal form of this assertion, which works in exactly the same way.

— **expanding subsets**

For every $\epsilon > 0$, define P_ϵ to be the subset of Teichmüller space consisting of hypebolic structures such that the set of geodesics whose length is within ϵ of the shortest length fill up the surface. The closure of P_ϵ maps to a compact subset of the modular space (Teichmüller space modulo the action of diffeomorphisms of the surface), since there is a lower bound to the length of the shortest geodesic on the surface in terms of epsilon. It follows that the P_ϵ form a neighborhood basis for P, since their intersection is P.

For each finite set of simple closed curves Γ which does not fill up the surface, we can choose in a natural way a vector field X_Γ on Teichmüller space which increases all of their lengths. More specifically, let us choose an equivariant Riemannian metric, and then define X_Γ to be the vector of unit length such that the sum of the logarithms of the derivatives of the elements of Γ in the direction X_Γ is maximized. (This method has the advantage of being easy to say; if one wished to actually perform the isotopy on a computer, there would be more efficient ways.)

A discontinuous vector field can be defined on Teichmüller space minus P by using X_Γ wherever Γ is the set of vectors of minimal length. We will approximate X_Γ (in a certain sense) by a continuous vector field, using a partition of unity to pass between the choices for different Γ.

The approximation depends on a parameter ϵ. Given ϵ, define the set U_Γ for each finite collection of simple curves Γ to consist of all hyperbolic structures such that Γ is exactly the set of simple closed geodesics whose length is less than $L + |\Gamma|\epsilon$, where L is the shortest length of a geodesic on the surface, and $|\Gamma|$ denotes the cardinality of Γ.

If ϵ is small enough, then the U_Γ form a covering of Teichmüller space. Indeed, there is a universal upper bound B to the number of geodesics of length less than say $L + 1$. Consider any $\epsilon < 1/B$. Enumerate all geodesics of length less than $L + 1$ in an order of non-decreasing length. If there is a first i such that the length of the ith is greater than $L + \epsilon i$, then $i > 1$ and the first $i - 1$ geodesics satisfy the condition. Otherwise, the set of all geodesics of length less than $L + 1$ satisfy the condition.

The covering by the U_Γ satisfies another property: if two sets U_Γ and U_{Γ}' intersect, then either $\Gamma \subset \Gamma$ or $\Gamma \subset \Gamma$. This is clear from the definition.

Choose a partition of unity $\{\lambda_\Gamma\}$ supported on the U_Γ. This can be done in a natural way by using only the sequence of lengths of geodesics on the surface in the definition of the partition functions. Define the vector field X_ϵ to be the average of the X_Γ with respect to the λ_Γ. To make this definition complete, we also have to define X_Γ when Γ fills the surface; define it to be identically zero.

If x is any point in Teichmüller space which is in the support of some λ_Γ where Γ does not fill up the surface, then X_ϵ cannot vanish at x. Indeed, for any curve γ in the intersection of all such Γ, each of the X_Γ increases its length, so the convex combination also increases its length. In particular, X_ϵ is never zero outside of $P_{B\,\epsilon}$.

The vector field X_ϵ generates a flow defined for all time, since the set of structures with shortest geodesic of length greater than a given constant is invariant by the flow and projects to a compact set in the modular space. Every compact set is eventually carried inside $P_{B\,\epsilon}$.

To get an isotopy as promised, the flows coming from various X_ϵ can be combined, using one epsilon for a while, then using a smaller epsilon, etc.

One complaint about this isotopy might be that it does not respect the stratification of Teichmüller space by sets of shortest geodesics. Actually, there is no isotopy which respects the stratification. The reason is simple. If γ_1 and γ_2 are any two distinct geodesics of shortest length on a closed surface, they can intersect in at most one point – if they were to intersect in two or more points, a cut and paste construction would produce a shorter geodesic. Consequently, a set of shortest geodesics which fills the surface cannot have any separating curves. On the other hand, it is easy to construct a hyperbolic structure on a surface of genus 2 or more where the shortest geodesic is a separating curve. Obviously, such a curve cannot remain shortest through the course of the isotopy.

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