Exercices and open questions to guide the discussion session on the course on TQFTs at the school on Quantum topology at VIASM, June 2025 by F. Costantino

Let $q = i = \sqrt{-1} \in \mathbb{C}$ and $U_q(\mathfrak{sl}_2)$ be the Hopf algebra generated by $K^{\pm 1}, E, F$ with relations:

$$KK^{-1} = K^{-1}K = 1, KE = q^2 EK, KF = q^{-2} FK, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, K^4 = 1, E^2 = F^2 = 0$$

Its bialgebra structure is given by :

$$\Delta(K) = K \otimes K, \Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

Its antipode and counit are :

$$\epsilon(K) = 1, \epsilon(E) = \epsilon(F) = 0, S(K) = K^{-1}, S(E) = -K^{-1}E, S(F) = -FK.$$

Exercise 9.31. Check that $U_q(\mathfrak{sl}_2)$ is a Hopf algebra.

Exercise 9.32. Let S be the two dimensional module generated by vectors $v_\pm$ such that $Kv_\pm = q^\pm v_\pm$, $Ev_+ = 0$ $Fv_+ = v_-$. Compute $E(v_-)$ and $F(v_-)$. Show that S is self dual, i.e. there is an isomorphism $D : S \to S^*$.

Exercise 9.33. With reference to the previous exercice, let $P = S \otimes S$. Show that P has a two dimensional subspace of vectors with K-eigenvalue equal to 1. Show that there are non-zero $U_q(sl_2)$-module morphisms $\epsilon : P \to \mathbb{C}$ and $\eta : \mathbb{C} \to P$ (where $\mathbb{C}$ is the trivial module via the counit of $U_q(sl_2)$). Show that $\epsilon \circ \eta = 0$ and let $n = \eta \circ \epsilon$. Show that $n \neq 0$ and $n^2 = 0$.

By the previous two exercices, we can realise in a graphical way the morphisms η and ϵ very easily by depicting P as $S \otimes S^*$ so that they become cup and caps.

Exercise 9.34. Let σ be the 1-dimensional module over $U_q(\mathfrak{sl}_2)$ such that K acts as -1 and E, F as 0. Show that $\sigma \otimes \sigma \simeq \mathbb{C}$. For each $X, Y \in \{S, P, S \otimes \sigma, P \otimes \sigma\}$ prove $X \otimes \sigma \simeq \sigma \otimes X$ and understand the structure of $Hom(X, Y)$ for $X, Y \in \{S, P, S \otimes \sigma, P \otimes \sigma\}$.

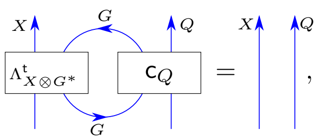
Also, it turns out that $S, S \otimes \sigma$, P and $P \otimes \sigma$ are all the projective indecomposable modules over $U_q(\mathfrak{sl}_2)$ (let's accept this for the time being). Therefore a generator for the ideal of projective modules over $U_q(\mathfrak{sl}_2)$ is $G = (S \oplus P) \otimes (1 \oplus \sigma)$.

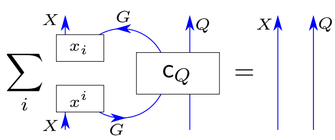
Exercise 9.35. Show that the trivial module $\mathbb{C}$ is **NOT** projective. (Hint: consider $\epsilon : P \to \mathbb{C}$. If $\mathbb{C}$ were projective then it would split...)

Exercise 9.36. The modified trace on the ideal of projective modules is entirely specified by stating that $mtr(Id_S) = 1$. Compute $mtr(Id_P)$ and $mtr(n)$, where $n : P \to P$ was defined above.

YOU CAN JUMP THE NEXT EXERCISE IF YOU LIKE: IT IS NEEDED ONLY TO COMPUTE THE TQFT MAPS. Given a projective module Q, let $x_i \in Hom(\mathbb{C}, Q)$ be a basis and $x^i \in Hom(Q, \mathbb{C})$ be the dual one wrt to the pairing given by the modified trace : $mtr(x_i \circ x^j) = \delta_{i,j}$. Let then $\Lambda^t_Q = \sum_i x_i \circ x^i$.

Exercise 9.37. Describe the cutting (Λ^t_Q in eq. (1)) and chromatic ($c_Q : G \otimes Q \to G \otimes Q$ here below or else see eq. (3)) morphisms for $Q = S, P$ via graphical calculus.

c_Q is such that $\forall X$: 

, or more explicitly 

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Exercise 9.38. Compute a system of generators of $\mathscr{S}(D^2)$ and $\mathscr{S}(S^1 \times [-1,1])$.

Here is a question of which I don't know the answer but should be easy :

Question 9.39. *What is a basis of* $\mathscr{S}(S^1 \times [-1,1])$*?*

From now on is a list of harder questions of which I don't know the answers and it would be good to try !

Question 9.40. *What is a basis of* $\mathscr{S}(S^1 \times S^1 \setminus D^2)$*? What about* $\mathscr{S}(S^1 \times S^1)$*?*

Remember : $\mathscr{S}(S)$ is the vector space of the $(3,2)$-TQFT we build... so if you solved the previous question the following becomes accessible and interesting:

Question 9.41. *Give the action of the self-diffeomorphisms of the torus* $S^1 \times S^1$ *consisting respectively of a* $\frac{\pi}{2}$*-rotation (a.k.a. "S"-matrix) and of a Dehn twist along* $S^1 \times \{pt\}$ *(a.k.a. "T"). (The result should satisfy the relations of* $SL(2;\mathbb{Z})$ *i.e.* $S^4 = Id$ *and* $(ST)^3 = S^2$*.)*

Question 9.42. *Consider the solid torus (minus a ball) as a cobordism from* $S^1 \otimes S^1$ *to* S^2*. Describe its action as a linear map on the skein modules via the move given by glueing a* 2*-handle (i.e. using the cutting map).*

Question 9.43. *Consider the solid torus (minus a ball) as a cobordism from* S^2 *to* $S^1 \otimes S^1$*. Describe its action as a linear map on the skein modules via the move given by glueing a* 1*-handle (i.e. using the coloring map).*

If you answered the previous questions, can you answer the following ?

Question 9.44. *What are the non semi-simple Turaev-Viro invariants of the Lens spaces* $L(p,q)$*?*

I expect that the answer should be related to the Reidemeister torsion...

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