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LMI-based conditions for finite-time stability of singular large-scale time-delay neural networks

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Abstract This paper investigates robust finite-time stability for a class of singular large-scale singular neural networks. The singular large-scale system under consideration is subjected to interconnected delays and bounded disturbances. Using the singular value theory and Lyapunov-Krasovskii function method, we propose new LMI-based criteria for the robust finite-time stability of such systems. The conditions are presented in terms of tractable linear matrix inequalities (LMIs), which can be solved efficiently by the LMI toolbox algorithm. A numerical example is given to indicate significant improvements of the proposed method.

Keywords: Stability, Singularity, Neural networks, LMIs, Interconnected delays, Lyapunov method.

1 Introduction

Stability analysis for large-scale neural networks (LSNNs), which is one of the most important topics in the qualitative theory of dynamical systems, has received considerable attention over the past decades (see, e.g., [1-3] and the references therein). A significant stability study of LSNNs focuses on the systems with delays. However, most of the existing results on LSNNs have concentrated on Lyapunov asymptotic stability (LAS), which is defined over an infinite time interval. the author of [4] introduced the concept of finite-time stability (FTS), which focuses on the transient behaviour of a system response over finite time interval. There have been a lots of interesting resuts on the FTS [5-7]. In the past years, in the context of large-scale systems problem of stability and control has been widely studied and found many significant applications [8, 9]. Especially, for large-scale neural networks, which incorporate numerous subsystems with enormous of numbers variables and contain interconnected delays, turn out to be increasingly more complicated due to its high dimension and structure distribution characteristics [10-13]. On the other hand, singular

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systems (also known as descriptor, implicit, differential-algebraic or general state-space systems) have attracted particular interest and many significant results in this area have been obtained [14-17]. For large-scale equations with delays, by using the Lyapunov function method combined with the LMI technique, some results on FTS were proposed by [18-21], however, the singularity was not considered there. For the singular large-scale equations, using singular value decomposition and the Lyapunov function method, some delay-dependent sufficient conditions were proposed by [22-24], but the neural structure was not considered there. When neural structure appears in large-scale systems, some stability results were reported in [25, 26], however, no singular parameter was considered there. It should be noted that there exist few results on the FTS of singular LSNNs with delays. The reason is that the singular LSNNs describe nonlinear time-delay systems of high dimension with complicated delay parameters consisting of large-scale differential-algebraic equations. Stability analysis of singular LSNNs with interconnected delays is much more complicated and requires an extensive calculation to investigate the stability conditions. To the best of our ability, there are no results on the FTS of the singular LSNNs with delays in the literature. This is an essential and challenging subject not only in theory but also in practice.

The aim of this work is to provide sufficient conditions for robust FTS of linear singular LSNNs with interconnected delays. Different from the existing methods, we have presented an equivalent form for the system by decomposing singular matrix method and by constructing a kind of novel Lyapunov function. The method based on the singular value theory and Lyapunov function approach has been used to decompose the system to fast and slow subsystems, which results in an explicit representation of the fast variables in terms of the initial conditions and the slow variables.

Comparing with the existing results, our paper has the following novel features. (i) The innovation of research approach. In this paper, we attempt to develop Lyapunov function method combined with SVD approach to solving robust stability problem. The proposed approach is the first trial in investigating robust stability of singular LSNNs with interconnected delays. (ii) The difficulty and generalization of the research result. The main difficulties and drawbacks in stability analysis of singular LSNNs are the presence of interconnected delays and disturbances. Our system model describes a wider class of LSNNs, which subject to the interconnected delays and disturbances, which gives rise to the difficulty in the stability study due to limited research techniques. The contributions of our paper can be summarized as follows.

(i) Robust FTS analysis of large-scale systems in the existing papers reveals some restrictions: either the delays, the singular structure or the neural structure is not considered. In our study, the above restrictions are removed and the delays are interacted between subsystems.

(ii) Combined with the LMI technique [27], by creating a new enhanced Lyapunov functionals, a new set of sufficient conditions for robust FTS is provided.

(iii) Delay-dependent FTS conditions are established in the form of strict LMIs, which can be easily solved by using interior point algorithm [28]. In addition, a design procedure has also proposed for the robust FTS of the system.

(iv) Through a given numerical example, we verify the validity of the stability conditions.

The following is the paper's structure: Section 2 contains definitions and mathematical preliminaries needed for the next sections. Section 3 presents the main result on FTS of descriptor LSNNs with a numerical example and simulations.

Notations. $\mathbb{R}^+$ represents the space of real positive numbers ; $\mathbb{C}$ is the space of all complex numbers, $]\mathbb{R}^k$ stands for the $k-$ dimensional space; $\mathbb{R}^{a\times b}$ is the space of $(a \times b)$- matrices; $C([0,M],\mathbb{R}^k)$ represents the space of continuous functions on $[0,M]$; $L_2([0,M],\mathbb{R}^k)$ represents the space of $2-$integrable functions on $[0,M]$. A is a positive definite matrix ($A > 0$) if $(Az,z) > 0$ for all $z \neq 0$.

2 Preliminaries

Consider singular neural networks described in the following large-scale complex form

$$\begin{cases} E_i\dot{y}_i(t) & = -A_iy_i(t) + B_if_i\Big(y_i(t)\Big) + \sum\limits_{i\neq j, j=1}^{\mathcal{M}} C_{ij}g_j\Big(y_j(t-\delta_{ij})\Big) + D_iw_i(t), \quad t \geq 0, \\ y_i(t) & = \phi_i(t), \quad t \in [-\delta, 0], \end{cases} \tag{1}$$

where $0 < \delta_{ij} \leq \delta$; $i \neq j$; $i, j = 1, 2, ..., \mathcal{M}$; $y_i(t) = [y_1^i(t),\, y_2^i(t)\, ...,\, y_{n_i}^i(t)]^\top \in \mathbb{R}^{n_i}$ is the state of the $i-$th neural in the networks; E_i is singular: rank $E_i = r_i \leq n_i, i = 1, 2, ..., \mathcal{M}$; $w_i(t) \in \mathbb{R}^{p_i}$ is the disturbance; $A_i = diag\{a_1^i,\, a_2^i,\, \ldots, a_{n_i}^i\} \in \mathbb{R}^{n_i\times n_i}$ satisfying $a_l^i > 0, \forall l = 1, ..., n_i$; $B_i \in \mathbb{R}^{n_i\times n_i}$, $C_{ij} \in \mathbb{R}^{n_i\times n_j}$ denote the connection and the discretely delayed weights, respectively, $D_i \in \mathbb{R}^{n_i\times p_i}$ is constant matrix of appropriate dimensions; $f_i(y_i(t)) = [f_1^i(y_1^i(t)),\, ...\, f_{n_i}^i(y_{n_i}^i(t))]^\top$; $g_j(y_j(t-\delta)) = [g_1^j(y_1^j(t-\delta)),\, ...\, g_{n_j}^j(y_{n_j}^j(t-\delta))]^\top$; $\phi_i(.) \in C([-\delta, 0]; \mathbb{R}^{n_i})$; the disturbance $w_i(t)$ satisfies

$$\exists h > 0 : \max_{i=1,\ldots,\mathcal{M}} \Big\{ \sup_{t>0}\{w_i^\top(t)w_i(t)\}\Big\} \leq h. \tag{2}$$

Let us set

$$\begin{aligned} D =& diag\{D_1, \cdots, D_{\mathcal{M}}\}, B = diag\{B_1, \cdots, B_{\mathcal{M}}\}, A = diag\{A_1, \cdots, A_{\mathcal{M}}\} \\ y^\top(t) =& [y_1(t)^\top, \ldots, y_{\mathcal{M}}(t)^\top], f^\top(y(t)) = [f_1(y_1(t))^\top, \ldots, f_{\amalg}n_iM(y_{\mathcal{M}}(t))^\top], \\ R =& diag\{R_1, \cdots, R_{\mathcal{M}}\},\, w(t) = [w_1(t)^\top, \ldots, w_{\mathcal{M}}(t)^\top], \varphi^\top(t) = [\varphi_1(t)^\top, \ldots, \varphi_{\mathcal{M}}(t)^\top], \\ E =& diag\{E_1, \cdots, E_{\mathcal{M}}\}, g^\top(y(t-\delta_{ij})) = [g_1(y_1(t-\delta_{ij}))^\top, \ldots, g_{\mathcal{M}}(y_{\mathcal{M}}(t-\delta_{ij}))^\top], \\ \overline{C}_{ij} =& \begin{cases} C_{ij} & \text{in the line i and the column j,} \\ 0 & \text{on the other positions,} \end{cases} \end{aligned}$$

then, the system (1) is given in the form

$$\begin{cases} E\dot{y}(t) & = -Ay(t) + Bf\Big(y(t)\Big) + \sum\limits_{i=1}^{\mathcal{M}} \sum\limits_{j\neq i, j=1}^{\mathcal{M}} \overline{C}_{ij}g\Big(y(t-\delta_{ij})\Big) + Dw(t), \quad t \geq 0, \\ y(t) & = \phi(t), \quad t \in [-\delta, 0]. \end{cases} \tag{3}$$

The activation functions satisfy Lipschitz conditions:

$$|f_1^i(y)| \leq \eta_1^i|y|,\; ...,\; |f_{n_i}^i(y)| \leq \eta_{n_i}^i|y|, \tag{4}$$

and

$$|g_1^j(y)| \leq \gamma_1^j|y|,\; ...,\; |g_{n_j}^j(y)| \leq \gamma_{n_j}^j|y|, \tag{5}$$

for all $y \in \mathbb{R}$.

Definition 1 System (3) is (i) regular if $det(sE + A)$ is not equal to zero; (ii) impulse-free if $deg(\det(sE + A)) = rankE,\ s \in \mathbb{C}$.

Definition 2 For $c_1 > 0, c_2 > 0, L > 0$ and a symmetric matrix $R > 0$, system (1) is robustly finite-time stable w.r.t. (c_1, c_2, L, R) if it is impulse-free, regular and the following relation holds for all disturbances $w_i(t)$ satisfying (2):

$$\sup_{\tau\in[-\delta,0]} \{\phi^\top(\tau)R\phi(\tau)\} \leq c_1 \quad \rightarrow \quad y^\top(t)Ry(t) < c_2, \quad \forall t \in [0, L].$$

Lemma 1 *([27]) Given matrices P, Q; $R, Q = Q^\top, P = P^\top$, we have*

$$P + R^\top Q^{-1} R < 0 \Leftrightarrow \begin{bmatrix} P & R^\top \\ R & -Q \end{bmatrix} < 0.$$

Lemma 2 *For matrices $\mathbb{P}, \mathbb{L}, \mathbb{U}, \mathbb{G}, \mathbb{S}, \mathbb{T}$, where $\mathbb{P} = \mathbb{P}^\top, \mathbb{S} = \mathbb{S}^\top > 0$ and $\mathbb{G} = \mathbb{G}^\top > 0$, we have*

$$\begin{pmatrix} \mathbb{P} + \mathbb{U}^\top \mathbb{G}^{-1} \mathbb{U} & \mathbb{T}^\top \\ \mathbb{L} & -\mathbb{S} \end{pmatrix} < 0 \Leftrightarrow \begin{pmatrix} \mathbb{P} & \mathbb{L}^\top & \mathbb{U}^\top \\ \mathbb{L} & -\mathbb{S} & 0 \\ \mathbb{U} & 0 & -\mathbb{G} \end{pmatrix} < 0.$$

Proof. The proof of Lemma 2 is an easy consequence of the Lemma 1.

Lemma 3 *([29]) For a matrix $0 < G \in \mathbb{R}^n$, and two scalars k_1, k_2 satisfy $0 \le k_1 < k_2$ and a vector function $x : [k_1, k_2] \longrightarrow \mathbb{R}^n$ such that the intergrations concerned are well defined, we have*

$$\Big(\int_{k_1}^{k_2} x(s)ds \Big)^\top G \Big(\int_{k_1}^{k_2} x(s)ds \Big) \le (k_2 - k_1) \int_{k_1}^{k_2} x^\top(s) G x(s) ds.$$

3 Robust stability

In this section, we provide new LMI criteria for FTS of system (1). Because of rank $E_i = r_i < n_i$, without lost of generality as proposed in [17], we assume that the matrix E_i has the form $E_i = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix}$ and denote

$$A_i = diag\{a_1^i,\, a_2^i,\, \ldots, a_{n_i}^i\} = \begin{pmatrix} \bar{A}_{11}^i & (0 \\ 0 & \bar{A}_{22}^i \end{pmatrix};\ \bar{A}_{11}^i = diag\{a_1^i,\ \ldots, a_{r_i}^i\} \in \mathbb{R}^{r_i \times r_i};$$

$$\bar{A}_{22}^i = diag\{a_{r_i+1}^i,\ \ldots, a_{n_i}^i\} \in \mathbb{R}^{(n_i - r_i)\times(n_i - r_i)};\ P_i = \begin{pmatrix} P_{11}^i & P_{12}^i \\ P_{21}^i & P_{22}^i \end{pmatrix};\ D_i = \begin{pmatrix} D_I^i \\ D_{II}^i \end{pmatrix};$$

$$f_i(y_i(t)) = \begin{pmatrix} f_I^i(.) \\ f_{II}^i(.) \end{pmatrix} \text{for all} \quad i, j = \overline{1, \mathcal{M}};\ B_i = \begin{pmatrix} \bar{B}_{11}^i & \bar{B}_{12}^i \\ \bar{B}_{21}^i & \bar{B}_{22}^i \end{pmatrix};\ C_{ij} = \begin{pmatrix} C_{11}^{ij} & C_{12}^{ij} \\ C_{21}^{ij} & C_{22}^{ij} \end{pmatrix};$$

$$g_j(y_j(t - \delta_{ij})) = \begin{pmatrix} g_I^j(.) \\ g_{II}^j(.) \end{pmatrix} \text{for all} \quad i, j = \overline{1, \mathcal{M}};\ D_I^i \in \mathbb{R}^{r_i \times p_i};\ \bar{B}_{11}^i,\ P_{11}^i \in \mathbb{R}^{r_i \times r_i},$$

$$C_{11}^{ij} \in \mathbb{R}^{r_i \times r_j};\ C_{12}^{ij} \in \mathbb{R}^{r_i \times (n_j - r_j)};\ C_{21}^{ij} \in \mathbb{R}^{(n_i - r_i)\times r_j};$$

$$f_I^i(.) = [f_1^i(y_1^i(t)),\ \ldots,\ f_{r_i}^i(y_{r_i}^i(t))]^\top, g_I^j(.) = [g_1^j(y_1^j(t - \delta_{ij})),\ \ldots,\ g^j r_j(y_{r_j}^j(t - \delta_{ij}))]^\top;$$

$$y_i(t) = \begin{pmatrix} y_I^i(t) \\ y_{II}^i(t) \end{pmatrix};\ y_I^i(t) = [y_1^i(t),\ \ldots,\ y_{r_i}^i(t)]^\top \in \mathbb{R}^{r_i};\ y_{II}^i(t) \in \mathbb{R}^{n_i - r_i}.$$

The system (1) is reduced to the following slow-fast subsystem

$$\begin{cases} \dot{y}_I^i(t) &= -\bar{A}_{11}^i y_I^i(t) + \bar{B}_{11}^i f_I^i(.) + \bar{B}_{12}^i f_{II}^i(.) + \sum\limits_{j=1, j\neq i}^{\mathcal{M}} \Big[C_{11}^{ij} g_I^j(.) + C_{12}^{ij} g_{II}^j(.) \Big] + D_I^i \omega_i(t), \\ 0 &= -\bar{A}_{22}^i y_{II}^i(t) + \bar{B}_{21}^i f_I^i(.) + \bar{B}_{22}^i f_{II}^i(.) + \sum\limits_{j=1, j\neq i}^{\mathcal{M}} \Big[C_{21}^{ij} g_I^j(.) + C_{22}^{ij} g_{II}^j(.) \Big] + D_{II}^i \omega_i(t), \\ y_i(t) &= \varphi_i(t), \quad t \in [-\delta, 0]. \end{cases} \tag{6}$$

The following notations are introduced for simplicity.

$$U_i = diag\{\eta_1^i, \eta_2^i, \ldots, \eta_{n_i}^i\};\ V_i = diag\{\gamma_1^i, \gamma_2^i, \ldots, \gamma_{n_i}^i\},$$

$$\begin{aligned}
\nu^i_{1,1} =& - P_iA_i - A_i^\top P_i^\top + (\mathcal{M}-1)Q_i + 3U_i^\top Z_iU_i - X_iA_i - A_i^\top X_i^\top + \sum_{j=1,i\neq j}^{\mathcal{M}} \delta_{ji}Y_i,\\
\nu^i_{j,j} =& - Q_j + 3V_j^\top Z_jV_j,\ j>i;\ \nu^i_{j,j} = -Q_{j-1} + 3V_{j-1}^\top Z_{j-1}V_{j-1},\ i\geq j;\ j=2,..,\mathcal{M},\\
\nu^i_{1,\mathcal{M}+1} =& - A_i^\top S_i^\top - X_i;\ \nu^i_{\mathcal{M}+1,\mathcal{M}+1} = -S_i - S_i^\top,\\
\nu^i_{\mathcal{M}+j,\mathcal{M}+j} =& - \delta_{ij}^{-1}Y_j,\ \forall j>i;\ \nu^i_{\mathcal{M}+j,\mathcal{M}+j} = -\delta^{-1}_{i,j-1}Y_{j-1},\ j\leq i;\ j=\overline{2,\mathcal{M}},\\
\nu^i_{1,2\mathcal{M}+1} =& P_iB_i;\ \nu^i_{2\mathcal{M}+1,2\mathcal{M}+1} = -Z_i;\ \nu^i_{1,2\mathcal{M}+j} = P_iC_{ij},\ j>i,\\
\nu^i_{1,2\mathcal{M}+j} =& P_iC_{i,j-1},\ i\geq j,\ j=2,...,\mathcal{M},\\
\nu^i_{2\mathcal{M}+j,2\mathcal{M}+j} =& - Z_j,\ i<j;\ \nu^i_{2\mathcal{M}+j,2\mathcal{M}+j} = -Z_{j-1},\ j=\overline{2,\mathcal{M}}, i\neq j;\ \nu^i_{1,3\mathcal{M}+1} = P_iD_i,\\
\nu^i_{3\mathcal{M}+1,3\mathcal{M}+1} =& - I; \nu^i_{1,3\mathcal{M}+j} = X_iC_{ij},\ j>i;\ \nu^i_{1,3\mathcal{M}+j} = X_iC_{i,j-1},\ j=\overline{2,\mathcal{M}}, i\geq j,\\
\nu^i_{3\mathcal{M}+j,3\mathcal{M}+j} =& - Z_j,\ j>i;\ \nu^i_{3\mathcal{M}+j,3\mathcal{M}+j} = -Z_{j-1},\ j=\overline{2,\mathcal{M}}, i\geq j,\\
\nu^i_{1,4\mathcal{M}+1} =& X_iB_i;\ \nu^i_{4\mathcal{M}+1,\mathcal{M}+1} = -Z_i,\\
\nu^i_{\mathcal{M}+1,4\mathcal{M}+j} =& S_iC_{ij},\ j>i;\ \nu^i_{\mathcal{M}+1,4\mathcal{M}+j} = S_iC_{i,j-1},\ j\leq i,\ j=\overline{2,\mathcal{M}},\\
\nu^i_{4\mathcal{M}+j,4\mathcal{M}+j} =& - Z_j,\ i<j;\ \nu^i_{4\mathcal{M}+j,4\mathcal{M}+j} = -Z_{j-1},\ j\leq i,\ j=\overline{2,\mathcal{M}},\\
\nu^i_{1,5\mathcal{M}+1} =& X_iD_i;\ \nu^i_{5\mathcal{M}+1,5\mathcal{M}+1} = -I, \nu^i_{\mathcal{M}+1,5\mathcal{M}+2} = S_iB_i;\ \nu^i_{5\mathcal{M}+2,5\mathcal{M}+2} = -Z_i,\\
\nu^i_{\mathcal{M}+1,5\mathcal{M}+3} =& S_iD_i;\ \nu^i_{5\mathcal{M}+3,5\mathcal{M}+3} = -I;\ \nu^i_{lk} = 0 \text{ for all other cases},\\
\vartheta_3 =& \min_{i=1,...,\mathcal{M}}\{\lambda_{min}(P^i_{11})\};\ \vartheta = \max_{i=1,...,\mathcal{M}}\left\{\frac{1}{\lambda_{min}(R_i)}\right\},\\
\vartheta_1 =& \max_{i=1,...,\mathcal{M}}\left\{\frac{\lambda_{max}(P_iE_i)}{\lambda_{min}(R_i)}\right\} + \delta(\mathcal{M}-1)\max_{i=1,...,\mathcal{M}}\left\{\frac{\lambda_{max}(Q_i}{\lambda_{min}(R_i)}\right\}\\
&+ (\mathcal{M}-1)\frac{\delta^2}{2}\max_{i=1,...,\mathcal{M}}\left\{\frac{\lambda_{max}(Y_i)}{\lambda_{min}(R_i)}\right\},\\
\vartheta_2 =& \max_{i=1,...,\mathcal{M}}\Big\{\lambda_{max}(R_i)\Big\};\ \vartheta_4 = \frac{\vartheta_1c_1 + 3\mathcal{M}hL}{\vartheta_3};\ \delta^1 = \min_{i,j=1,...,\mathcal{M};i\neq j}\{\delta_{ij}\},\\
\lambda_1 =& (2\mathcal{M}+1)\max_{i=1,...,\mathcal{M}}\{\parallel [\bar{A}^i_{22}]^{-1}\bar{B}^i_{21}\parallel^2; \parallel [\bar{A}^i_{22}]^{-1}\bar{B}^i_{22}\parallel^2\},\\
\lambda_2 =& (2\mathcal{M}+1)\max_{i\neq j;i,j=1,...,\mathcal{M}}\{\parallel [\bar{A}^i_{22}]^{-1}C^{ij}_{21}\parallel^2; \parallel [\bar{A}^i_{22}]^{-1}C^{ij}_{22}\parallel^2\},\\
\lambda_3 =& (2\mathcal{M}+1)\max_{i=1,...,\mathcal{M}}\{\parallel [\bar{A}^i_{22}]^{-1}D^i_2\parallel^2\};\ \beta_0 = 1 - \lambda_1\max_{i=1,...,\mathcal{M}}\{\parallel U_i\parallel^2\},\\
\beta_0\beta_1 =& \lambda_1\max_{i=1,...,\mathcal{M}}\{\parallel U_i\parallel^2\};\ \beta_0\beta_2 = \lambda_2\max_{i=1,...,\mathcal{M}}\{\parallel V_i\parallel^2\};\ \beta_0\beta_3 = \lambda_3,\\
\vartheta_5 =& \sum_{l=0}^{[\frac{L}{\delta^1}]}\Big[\beta_2(\mathcal{M}-1)\Big]^l;\ \vartheta_6 = 1 + \beta_1\vartheta_5 + (\mathcal{M}-1)\beta_2\vartheta_5,\\
d(c_1) =& [2(\mathcal{M}-1)\vartheta c_1\beta_2 + \mathcal{M}h\beta_3]\vartheta_5.
\end{aligned}$$

Theorem 1 *The singular LSNNs (1) is robustly finite-time stable w.r.t.* (c_1, c_2, L, R) *if there exist non-singular matrices* P_i*, symmetric matrices* $Y_i > 0$, $S_i > 0$, $Q_i > 0$*, diagonal matrices* $Z_i > 0$*, matrices* X_i, $i = 1, ..., \mathcal{M}$ *and a scalar* $\beta > 0$ *such that*

$$P_iE_i = E_i^\top P_i^\top \geq 0; \tag{7}$$

$$\nu^i = \left(\nu^i_{lk}\right)_{(5\mathcal{M}+3)\times(5\mathcal{M}+3)} < 0, \quad i = 1, ..., \mathcal{M}; \tag{8}$$

$$\lambda_1\max_{i=1,...,\mathcal{M}}\{\parallel U_i\parallel^2\} - 1 < 0; \tag{9}$$

$$\vartheta_4\vartheta_6e^{\beta L} + d(c_1) < \frac{c_2}{\vartheta_2}. \tag{10}$$

Proof. *1. The regularity and impulse-free.* We first note that $(sE+A)` = diag\{sE_1+A_1,\ sE_2+A_2,\ ...,\ sE_{\mathcal{M}}+A_{\mathcal{M}}\}$ where $E_i = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix}$, $A_i = diag\{a_1^i,\ a_2^i,\ \dots,\ a_{n_i}^i\}$, then

$$\det(sE+A) = \det(sE_1+A_1)\det(sE_2+A_2)\det(sE_3+A_3)...\det(sE_{\mathcal{M}}+A_{\mathcal{M}}).$$

Moreover, we have

$$\begin{aligned}\det(sE_i+A_i) &= (s+a_1^i)(s+a_2^i)...(s+a_{r_i}^i)(a_{r_i+1}^i)...(a_{n_i}^i)\\ &= (a_{r_i+1}^i)...(a_{n_i}^i)\Big[s^{r_i}+\bar{a}_{r_i-1}^i s^{r_i-1}+...+\bar{a}_1^i s+\det(\bar{A}_{11}^i)\Big],\end{aligned}$$

where $a_l^i > 0$, $\forall i = \overline{1,\mathcal{M}}$; $l = \overline{1,n_i}$. Thus, $\det(sE_i+A_i)$ is not identical zero for all $i = \overline{1,\mathcal{M}}$. This implies that $\det(sE+A)$ is also not identical zero or neural network (1) is regular. Furthermore, we see that

$$\deg(\det(sE+A)) = \deg(\det(sE_1+A_1))+...+\deg(\det(sE_{\mathcal{M}}+A_{\mathcal{M}})) = r_1+...+r_{\mathcal{M}} = rankE.$$

Hence, the system (1) is impulse-free.

2. The robust FTS. We consider the Lyapunov-Krasovskii functionals:

$$\mathbb{V}(t,y_t) = \sum_{i=1}^{\mathcal{M}}\Big[\mathbb{V}_{i1}(t,y_t)+\mathbb{V}_{i2}(t,y_t)+\mathbb{V}_{i3}(t,y_t)\Big],$$

where

$$\begin{aligned}\mathbb{V}_{i1}(t,y_t) &= e^{\beta t}y_i(t)^\top P_iE_iy_i(t),\\ \mathbb{V}_{i2}(t,y_t) &= e^{\beta t}\sum_{j=1,\ j\neq i}^{\mathcal{M}}\int_{t-\delta_{ij}}^{t} y_j(s)^\top Q_jy_j(s)ds,\\ \mathbb{V}_{i3}(t,y_t) &= e^{\beta t}\sum_{j=1,\ j\neq i}^{\mathcal{M}}\int_{t-\delta_{ij}}^{t}\int_s^t y_j(v)^\top Y_jy_j(v)dvds.\end{aligned}$$

The derivative of $V(t,y_t)$ gives

$$\begin{aligned}\dot{\mathbb{V}}_{i1}(t,y_t) &= \beta\mathbb{V}_{i1}(t,y_t)+e^{\beta t}y_i^\top(t)[-A_i^\top P_i^\top-P_iA_i]y_i(t)+2e^{\beta t}y_i^\top(t)P_iB_if_i(y_i(t))\\ &\quad+2e^{\beta t}y_i^\top(t)P_i\sum_{i\neq j,j=1}^{\mathcal{M}}C_{ij}g_j(y_j(t-\delta_{ij}))+2e^{\beta t}y_i^\top(t)P_iD_iw_i(t);\\ \dot{\mathbb{V}}_{i2}(t,y_t) &= \beta\mathbb{V}_{i2}(t,y_t)+e^{\beta t}\sum_{i\neq j,j=1}^{\mathcal{M}}y_j^\top(t)Q_jy_j(t)-e^{\beta t}\sum_{j=1,j\neq i}^{\mathcal{M}}y_j^\top(t-\delta_{ij})Q_jy_j(t-\delta_{ij}).\\ \dot{\mathbb{V}}_{i3}(t,y_t) &= \beta\mathbb{V}_{i3}(t,y_t)+e^{\beta t}\sum_{i\neq j,j=1}^{\mathcal{M}}\delta_{ij}y_j^\top(t)Y_jy_j(t)-e^{\beta t}\sum_{j=1,j\neq i}^{\mathcal{M}}\int_{t-\delta_{ij}}^t y_j^\top(t)Y_jy_j(t).\end{aligned}$$

Based on (4), (5) and the following derived inequality estimations

$$\begin{aligned}&2y_i^\top(t)P_iB_if_i(y_i(t)) \le y_i^\top(t)P_iB_iZ_i^{-1}B_i^\top P_i^\top y_i(t)+y_i^\top(t)U_i^\top Z_iU_iy_i(t);\\ &2y_i^\top(t)P_i\sum_{i\neq j,j=1}^{\mathcal{M}}C_{ij}g_j(y_j(t-\delta_{ij})) \le \sum_{j=1,j\neq i}^{\mathcal{M}}y_j^\top(t-\delta_{ij})V_j^\top Z_jV_jy_j(t-\delta_{ij})\\ &\qquad+\sum_{j\neq i,j=1}^{\mathcal{M}}y_i^\top(t)P_iC_{ij}Z_j^{-1}C_{ij}^\top P_iy_i(t);\end{aligned}$$

$$2y_i^\top(t)P_iD_iw_i(t) \le y_i^\top(t)P_iD_iD_i^\top P_i^\top y_i(t) + w_i^\top(t)w_i(t);$$
$$-\delta_{ij}\int_{t-\delta_{ij}}^{t} y_j^\top(t)Y_jy_j(t) \le -\Big(\int_{t-\delta_{ij}}^{t} y_j(t)dt\Big)Y_j\Big(\int_{t-\delta_{ij}}^{t} y_j(t)dt\Big);$$
$$-2e^{\beta t}\dot{y}_i^\top(t)E_i^\top S_i\Big[E_i\dot{y}_i(t) + A_iy_i(t) - B_if_i\Big(y_i(t)\Big) - \sum_{j=1,j\neq i}^{\mathcal{M}} C_{ij}g_j\Big(y_j(t-\delta_{ij})\Big) - D_iw_i(t)\Big] = 0;$$
$$2e^{\beta t}\dot{y}_i^\top(t)E_i^\top S_iB_if_i\Big(y_i(t)\Big) \le [E_i\dot{y}_i(t)]^\top S_iB_iZ_i^{-1}B_i^\top S_i^\top[E_i\dot{y}_i(t)] + y_i(t)^\top U_i^\top Z_iU_iy_i(t);$$
$$2e^{\beta t}\dot{y}_i^\top(t)E_i^\top S_i\sum_{j=1,j\neq i}^{\mathcal{M}} C_{ij}g_j\Big(y_j(t-\delta_{ij})\Big) \le \sum_{j=1,j\neq i}^{\mathcal{M}}[E_i\dot{y}_i(t)]^\top S_iC_{ij}Z_j^{-1}C_{ij}^\top S_i^\top[E_i\dot{y}_i(t)] + \sum_{j\neq i,j=1}^{\mathcal{M}} y_j^\top(t-\delta_{ij})V_j^\top Z_jV_jy_j(t-\delta_{ij});$$
$$2e^{\beta t}\dot{y}_i^\top(t)E_i^\top S_iD_iw_i(t) \le [E_i\dot{y}_i(t)]^\top S_iD_iD_i^\top S_i^\top[E_i\dot{y}_i(t)] + w_i(t)^\top w_i(t);$$
$$-2e^{\beta t}y_i(t)^\top X_i\Big[E_i\dot{y}_i(t) + A_iy_i(t) - B_if_i\Big(y_i(t)\Big) - \sum_{j\neq i,j=1}^{\mathcal{M}} C_{ij}g_j\Big(y_j(t-\delta_{ij})\Big) - D_iw_i(t)\Big] = 0;$$
$$2y_i(t)^\top X_iB_if_i\Big(y_i(t)\Big) \le y_i^\top(t)X_iB_iZ_i^{-1}B_i^\top X_i^\top y_i(t) + y_i^\top(t)U_i^\top Z_iU_iy_i(t);$$
$$2y_i^\top(t)X_i\sum_{j\neq i,j=1}^{\mathcal{M}} C_{ij}g_j\Big(y_j(t-\delta_{ij})\Big) \le \sum_{i\neq j,j=1}^{\mathcal{M}} y_i^\top(t)X_iC_{ij}Z_j^{-1}C_{ij}^\top X_i^\top y_j(t) + \sum_{j\neq i,j=1}^{\mathcal{M}} y_j^\top(t-\delta_{ij})V_j^\top Z_jV_jy_j(t-\delta_{ij});$$
$$2y_i^\top(t)X_iD_iw_i(t) \le y_i^\top(t)(t)X_iD_iD_i^\top X_i^\top y_i(t) + w_i(t)^\top w_i(t),$$

we get

$$\begin{aligned}
\dot{\mathbb{V}}(t,y_t) - \beta\mathbb{V}(t,y_t) \le{}& e^{\beta t}\sum_{i=1}^{\mathcal{M}} y_i^\top(t)\Big[-P_iA_i - A_i^\top P_i^\top + P_iB_iZ_i^{-1}B_i^\top P_i^\top + (\mathcal{M}-1)Q_i \\
&+ 3U_i^\top Z_iU_i + \sum_{i\neq j,j=1}^{\mathcal{M}} P_iC_{ij}Z_j^{-1}C_{ij}^\top P_i + P_iD_iD_i^\top P_i^\top + \sum_{j\neq i,j=1}^{\mathcal{M}}\delta_{ji}Y_i - X_iA_i \\
&- A_i^\top X_i^\top + X_iB_iZ_i^{-1}B_i^\top X_i^\top + \sum_{j\neq i,j=1}^{\mathcal{M}} X_iC_{ij}Z_j^{-1}C_{ij}^\top X_i + X_iD_iD_i^\top X_i^\top\Big]y_i(t) \\
&+ e^{\beta t}\sum_{i=1}^{\mathcal{M}}\sum_{j\neq i,j=1}^{\mathcal{M}} y_j^\top(t-\delta_{ij})\Big[-Q_j + 3V_j^\top Z_jV_j\Big]y_j(t-\delta_{ij}) \\
&- e^{\beta t}\sum_{i=1}^{\mathcal{M}}\sum_{j\neq i,j=1}^{\mathcal{M}}\Big(\int_{t-\delta_{ij}}^{t} y_j(t)dt\Big)[\delta_{ij}^{-1}Y_j]\Big(\int_{t-\delta_{ij}}^{t} y_j(t)dt\Big) \\
&- 2e^{\beta t}\sum_{i=1}^{\mathcal{M}}[E_i\dot{y}_i(t)]^\top\Big(-S_i - S_i^\top + S_iB_iZ_i^{-1}B_i^\top S_i^\top + \sum_{j\neq i,j=1}^{\mathcal{M}} S_iC_{ij}Z_j^{-1}C_{ij}^\top S_i^\top \\
&+ S_iD_iD_i^\top S_i^\top\Big)[E_i\dot{y}_i(t)] + 3e^{\beta t}\sum_{i=1}^{\mathcal{M}} w_i^\top(t)w_i(t) + 2e^{\beta t}y_i^\top(t)[-A_i^\top S_i^\top - X_i][E_i\dot{y}_i(t)]
\end{aligned}$$

$$\leq e^{\beta t}\sum_{i=1}^{\mathcal{M}}\left([y_i(t)]^\top\,[z_i^2]^\top\,...\,[z_i^{\mathcal{M}}]^\top\,[E_i\dot{y}_i(t)]^\top\,[z_i^{\mathcal{M}+2}]^\top\,...\,[z_i^{2\mathcal{M}}]^\top\right)\begin{pmatrix}\Phi^i & 0\\ 0 & \Gamma^i\end{pmatrix}\begin{pmatrix}y_i(t)\\ z_i^2\\ ...\\ z_i^{\mathcal{M}}\\ E_i\dot{y}_i(t)\\ z_i^{\mathcal{M}+2}\\ ...\\ z_i^{2\mathcal{M}}\end{pmatrix}$$

$$+3e^{\beta t}\sum_{i=1}^{\mathcal{M}}w_i^\top(t)w_i(t),$$

where $z_i^j = y_j(t-\delta_{ij})$ if $j > i$; $z_i^j = y_{j-1}(t-\delta_{i,j-1})$ if $i \geq j$; $z_i^{\mathcal{M}+j} = \int_{t-\delta_{ij}}^t y_j(t)dt$ if $j > i$; $z_i^{\mathcal{M}+j} = \int_{t-\delta_{i,j-1}}^t y_{j-1}(t)dt$ if $j \leq i$, $\forall i = \overline{1,\mathcal{M}}$, $\forall j = \overline{2,\mathcal{M}}$, and $\Gamma^i = diag\{\Gamma^i_{\mathcal{M}+2},...,\Gamma^i_{2\mathcal{M}}\}$; $\Phi^i = \left(\Phi^i_{kl}\right)_{(\mathcal{M}+1)\times(\mathcal{M}+1)}$, $i = \overline{1,\mathcal{M}}$, in which

$$\begin{aligned}
\Phi^i_{1,1} =& -P_iA_i - A_i^\top P_i^\top + P_iB_iZ_i^{-1}B_i^\top P_i^\top + (\mathcal{M}-1)Q_i + 3U_i^\top Z_iU_i - X_iA_i - A_i^\top X_i^\top\\
&+ P_iD_iD_i^\top P_i^\top + \sum_{j\neq i,j=1}^{\mathcal{M}}\delta_{ji}Y_i + \sum_{j\neq i,j=1}^{\mathcal{M}}P_iC_{ij}Z_j^{-1}C_{ij}^\top P_i + X_iB_iZ_i^{-1}B_i^\top X_i^\top\\
&+ \sum_{j\neq i,j=1}^{\mathcal{M}}X_iC_{ij}Z_j^{-1}C_{ij}^\top X_i + X_iD_iD_i^\top X_i^\top;\\
\Phi^i_{1,\mathcal{M}+1} =& -A_i^\top S_i^\top - X_i;\\
\Phi^i_{j,j} =& -Q_j + 3V_j^\top Z_jV_j \quad \text{if } j > i;\ \Phi^i_{j,j} = -Q_{j-1} + 3V_{j-1}^\top Z_{j-1}V_{j-1} \quad \text{if } j \leq i;\ j = \overline{2,\mathcal{M}};\\
\Phi^i_{\mathcal{M}+1,\mathcal{M}+1} =& -S_i - S_i^\top + S_iB_iZ_i^{-1}B_i^\top S_i^\top + \sum_{j\neq i,j=1}^{\mathcal{M}}S_iC_{ij}Z_j^{-1}C_{ij}^\top S_i^\top + S_iD_iD_i^\top S_i^\top;\ j = 2,...,\mathcal{M};\\
\Phi^i_{j,k} =& 0 \quad \text{for all other cases;}\\
\Gamma^i_{\mathcal{M}+j,\mathcal{M}+j} =& -\delta_{ij}^{-1}Y_j,\ \text{if } j > i;\ \Gamma^i_{\mathcal{M}+j,\mathcal{M}+j} = -\delta_{i,j-1}^{-1}Y_{j-1},\ \text{if } j \leq i;\ j = 2,...,\mathcal{M};
\end{aligned}$$

Applying Lemma 1, the condition (8) is equivalent to $\begin{pmatrix}\Phi^i & 0\\ 0 & \Gamma^i\end{pmatrix} < 0$, $i = 1,2,...,\mathcal{M}$, and using condition (2), we obtain

$$\dot{\mathbb{V}}(t,y_t) - \beta\mathbb{V}(t,y_t) \leq 3\mathcal{M}he^{\beta t}. \tag{11}$$

We get by integrating both sides of (11) from 0 to t

$$\mathbb{V}(t,y_t) \leq \Big(\mathbb{V}(0,y_0) + 3\mathcal{M}Lh\Big)e^{\beta L}, \quad t \in [0,L]. \tag{12}$$

Furthermore, we have

$$
\begin{aligned}
\mathbb{V}(0,y_0) &\le \sum_{i=1}^{\mathcal{M}} \lambda_{max}(P_iE_i) y_i^\top(0) y_i(0) + \sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \delta_{ij}\lambda_{max}(Q_j) \sup_{s\in[-\delta,0]} \varphi_i^\top(s)\varphi_i(s) \\
&\quad + \sum_{i=1}^{\mathcal{M}} \sum_{j\neq i, j=1}^{\mathcal{M}} \frac{\delta_{ij}^2}{2} \lambda_{max}(Y_j) \sup_{s\in[-\delta,0]} \varphi_j^\top(s)\varphi_j(s) \\
&\le \sum_{i=1}^{\mathcal{M}} \frac{\lambda_{max}(P_iE_i)}{\lambda_{min}(R_i)} y_i^\top(0) R_i y_i(0) + \sum_{i=1}^{\mathcal{M}} (\mathcal{M}-1) \frac{\delta\lambda_{max}(Q_i)}{\lambda_{min}(R_i)} \sup_{s\in[-\delta,0]} \varphi_i^\top(s) R_i \varphi_i(s) \\
&\quad + (\mathcal{M}-1)\frac{\delta^2}{2} \sum_{i=1}^{\mathcal{M}} \frac{\lambda_{max}(Y_i)}{\lambda_{min}(R_i)} \sup_{s\in[-\delta,0]} \varphi_i^\top(s) R_i \varphi_i(s) \le \vartheta_1 c_1,
\end{aligned} \tag{13}
$$

which gives

$$
\mathbb{V}(t,y_t) \le (\vartheta_1 c_1 + 3\mathcal{M}Lh) e^{\beta L}. \tag{14}
$$

Moreover, we see that

$$
y(t)^\top R y(t) \le \max_{i=1,\mathcal{M}} \{\lambda_{\max}(R_i)\} \sum_{i=1}^{\mathcal{M}} y_i(t)^\top y_i(t) := \vartheta_2 \sum_{i=1}^{\mathcal{M}} \Big[\parallel y_I^i(t) \parallel^2 + \parallel y_{II}^i(t) \parallel^2 \Big],
$$

and from (7), we get $P_{21}^i = 0, P_{11}^i = [P_{11}^i] > 0$. Now, we will estimate the state solutions $\parallel y_I^i(t) \parallel^2$, $\parallel y_{II}^i(t) \parallel^2$. From the view of $\mathbb{V}(t,y_t)$ we have

$$
\mathbb{V}(t,y_t) \ge \sum_{i=1}^{\mathcal{M}} y_i(t)^\top P_iE_i y_i(t) = \sum_{i=1}^{\mathcal{M}} y_i(t)^\top \begin{pmatrix} P_{11}^i & 0 \\ 0 & 0 \end{pmatrix} y_i(t) \ge \sum_{i=1}^{\mathcal{M}} \lambda_{min}(P_{11}^i) \parallel y_I^i(t) \parallel^2 .
$$

This inequality with (14) gives

$$
\sum_{i=1}^{\mathcal{M}} \parallel y_I^i(t) \parallel^2 \le \frac{1}{\vartheta_3} e^{\beta L} \Big[\vartheta_1 c_1 + 3\mathcal{M}hL\Big] \le \frac{\vartheta_1 c_1 + 3\mathcal{M}hL}{\vartheta_3} e^{\beta L}, \quad \forall t \in [0,L]. \tag{15}
$$

Combine with (6), as a result of

$$
y_{II}^i(t) = [\bar{A}_{22}^i]^{-1}\bar{B}_{21}^i f_I^i(.) + [\bar{A}_{22}^i]^{-1}\bar{B}_{22}^i f_{II}^i(.) + \sum_{j\neq i, j=1}^{\mathcal{M}} [\bar{A}_{22}^i]^{-1}\Big[C_{21}^{ij} g_I^j(.) + C_{22}^{ij} g_{II}^j(.)\Big] + [\bar{A}_{22}^i]^{-1} D_{II}^i \omega_i(t),
$$

we have

$$
\begin{aligned}
\parallel y_{II}^i(t) \parallel^2 &\le (2\mathcal{M}+1) \parallel [\bar{A}_{22}^i]^{-1}\bar{B}_{21}^i \parallel^2 \parallel f_I^i(.) \parallel^2 + (2\mathcal{M}+1) \parallel [\bar{A}_{22}^i]^{-1}\bar{B}_{22}^i \parallel^2 \parallel f_{II}^i(.) \parallel^2 \\
&\quad + (2\mathcal{M}+1) \sum_{i\neq j, j=1}^{\mathcal{M}} \parallel [\bar{A}_{22}^i]^{-1} C_{21}^{ij} \parallel^2 \parallel g_I^j(.) \parallel^2 + (2\mathcal{M}+1) \sum_{i\neq j, j=1}^{\mathcal{M}} \parallel [\bar{A}_{22}^i]^{-1} C_{22}^{ij} \parallel^2 \parallel g_{II}^j(.) \parallel^2 \\
&\quad + (2\mathcal{M}+1) \parallel [\bar{A}_{22}^i]^{-1} D_{II}^i \parallel^2 \parallel \omega_i(t) \parallel^2 .
\end{aligned}
$$

Setting

$$
\begin{aligned}
\lambda_1 &= (2\mathcal{M}+1) \max_{i=1,\ldots,\mathcal{M}} \{\parallel [\bar{A}_{22}^i]^{-1}\bar{B}_{21}^i \parallel^2; \parallel [\bar{A}_{22}^i]^{-1}\bar{B}_{22}^i \parallel^2\}; \\
\lambda_2 &= (2\mathcal{M}+1) \max_{j\neq i, i,j=1,\ldots,\mathcal{M}} \{\parallel [\bar{A}_{22}^i]^{-1} C_{21}^{ij} \parallel^2; \parallel [\bar{A}_{22}^i]^{-1} C_{22}^{ij} \parallel^2\}; \\
\lambda_3 &= (2\mathcal{M}+1) \max_{i=1,\ldots,\mathcal{M}} \{\parallel [\bar{A}_{22}^i]^{-1} D_{II}^i \parallel^2\};
\end{aligned}
$$

we have

$$\begin{aligned}
\| y^i_{II}(t) \|^2 \leq & \lambda_1 \| f_i(.) \|^2 + \lambda_2 \sum_{i\neq j, j=1}^{\mathcal{M}} \| g_j(.) \|^2 + \lambda_3 \omega_i(t)\omega_i^\top(t) \\
\leq & \lambda_1 \| U_i \|^2 \| y_i(t) \|^2 + \lambda_2 \sum_{j\neq i, j=1}^{\mathcal{M}} \| V_j \|^2 \| y_j(t-\delta_{ij}) \|^2 + \lambda_3 \omega_i(t)\omega_i^\top(t) \\
\leq & \lambda_1 \max_{i=1,\dots,\mathcal{M}} \{\| U_i \|^2\} \sum_{i=1}^{\mathcal{M}} \| y^i_I(t) \|^2 + \lambda_1 \max_{i=1,\dots,\mathcal{M}} \{\| U_i \|^2\} \sum_{j=1}^{\mathcal{M}} \| y^j_{II}(t) \|^2 \\
& + \lambda_2 \max_{j=1,\dots,\mathcal{M}} \{\| V_j \|^2\} \sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \| y^j_I(t-\delta_{ij}) \|^2 \\
& + \lambda_2 \max_{j=1,\dots,\mathcal{M}} \{\| V_j \|^2\} \sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \| y^j_{II}(t-\delta_{ij}) \|^2 + \lambda_3 \sum_{j=1}^{\mathcal{M}} \omega_j(t)\omega_j^\top(t),
\end{aligned}$$

then

$$\begin{aligned}
\Big(1 - \lambda_1 \max_{i=1,\dots,\mathcal{M}} \{\| U_i \|^2\}\Big) \sum_{j=1}^{\mathcal{M}} \| y^j_{II}(t) \|^2 \leq & \lambda_1 \max_{j=1,\dots,\mathcal{M}} \{\| U_j \|^2\} \sum_{j=1}^{\mathcal{M}} \| y^j_I(t) \|^2 \\
& + \lambda_2 \max_{j=1,\dots,\mathcal{M}} \{\| V_j \|^2\} \sum_{i=1}^{\mathcal{M}} \sum_{j\neq i, j=1}^{\mathcal{M}} \| y^j_I(t-\delta_{ij}) \|^2 \\
& + \lambda_2 \max_{j=1,\dots,\mathcal{M}} \{\| V_j \|^2\} \sum_{i=1}^{\mathcal{M}} \sum_{j\neq i, j=1}^{\mathcal{M}} \| y^j_{II}(t-\delta_{ij}) \|^2 \\
& + \lambda_3 \sum_{j=1}^{\mathcal{M}} \omega_j(t)\omega_j^\top(t).
\end{aligned} \tag{16}$$

Since $\beta_0 = \Big(1 - \lambda_1 \max_{i=1,\dots,\mathcal{M}} \{\| U_i \|^2\}\Big) > 0$ and

$$\beta_1 = \frac{\lambda_1 \max_{i=1,\dots,\mathcal{M}} \{\| U_i \|^2\}}{\beta_0}; \ \beta_2 = \frac{\lambda_2 \max_{j=1,\dots,\mathcal{M}} \{\| V_j \|^2\}}{\beta_0}; \ \beta_3 = \frac{\lambda_3}{\beta_0},$$

the condition (16) implies

$$\begin{aligned}
\sum_{i=1}^{\mathcal{M}} \| y^i_{II}(t) \|^2 \leq & \beta_1 \sum_{i=1}^{\mathcal{M}} \| y^i_I(t) \|^2 + \beta_2 \sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \| y^j_I(t-\delta_{ij}) \|^2 \\
& + \beta_2 \sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \| y^j_{II}(t-\delta_{ij}) \|^2 + \beta_3 \sum_{j=1}^{\mathcal{M}} \| \omega_j(t) \|^2 .
\end{aligned} \tag{17}$$

• If $(t-\delta_{ij}) \in [-\delta, 0]$, we have

$$\begin{aligned}
\| y^j_I(t-\delta_{ij}) \|^2 \leq & \| y_j(t-\delta_{ij}) \|^2 = \varphi_j(t-\delta_{ij})^\top \varphi_j(t-\delta_{ij}) \\
\leq & \frac{1}{\lambda_{min}(R_j)} \varphi_j(t-\delta_{ij})^\top R_j \varphi_j(t-\delta_{ij}),
\end{aligned}$$

hence

$$\begin{aligned}
\sum_{i=1}^{\mathcal{M}} \sum_{j=1, j\neq i}^{\mathcal{M}} \| y^j_I(t-\delta_{ij}) \|^2 \leq & \max_{j=1,\dots,\mathcal{M}} \{\frac{1}{\lambda_{min}(R_j)}\} \sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \varphi_j(t-\delta_{ij})^\top R_j \varphi_j(t-\delta_{ij}) \\
:= & (\mathcal{M}-1)\vartheta c_1,
\end{aligned}$$

in which $\vartheta = \max_{j=1,\dots,\mathcal{M}}\{\frac{1}{\lambda_{min}(R_j)}\}$.

• If $(t-\delta_{ij}) \in [0, L]$, using (15) we have

$$\sum_{i=1}^{\mathcal{M}} \sum_{j\neq i, j=1}^{\mathcal{M}} \parallel y_I^j(t-\delta_{ij}) \parallel^2 \leq (\mathcal{M}-1)\frac{\vartheta_1 c_1 + 3\mathcal{M}hL}{\vartheta_3} e^{\beta L} := (\mathcal{M}-1)\vartheta_4 e^{\beta L},$$

and hence for $t \in [0, L]$ we obtain that

$$\sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \parallel y_I^j(t-\delta_{ij}) \parallel^2 \leq (\mathcal{M}-1)\Big[\vartheta c_1 + \vartheta_4 e^{\beta L}\Big].$$

Therefore, from (17) it follows the following estimation

$$\begin{aligned}\sum_{i=1}^{\mathcal{M}} \parallel y_{II}^i(t) \parallel^2 &\leq \beta_1\vartheta_4 e^{\beta L} + \beta_2(\mathcal{M}-1)\Big[\vartheta c_1 + \vartheta_4 e^{\beta L}\Big] + \beta_3\mathcal{M}h + \beta_2 \sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \parallel y_{II}^j(t-\delta_{ij}) \parallel^2 \\ &\leq a + \beta_2 \sum_{i=1}^{\mathcal{M}} \sum_{j\neq i, j=1}^{\mathcal{M}} \parallel y_{II}^j(t-\delta_{ij}) \parallel^2,\ t \in [0, L].\end{aligned}$$

We still to estimate the sum $\sum_{i=1}^{\mathcal{M}} \sum_{j\neq i, j=1}^{\mathcal{M}} \parallel y_{II}^j(t-\delta_{ij}) \parallel^2$ on $[0, L]$ as follows. Setting $\delta^1 = \min_{i\neq j; i,j=\overline{1,\mathcal{M}}}\{\delta_{ij}\}$.

a) Case $t \in [0, \delta^1] \Rightarrow t - \delta_{ij} \in [-\delta, 0]$, we get $\sum_{i=1}^{\mathcal{M}} \sum_{i\neq j, j=1}^{\mathcal{M}} \parallel y_{II}^j(t-\delta_{ij}) \parallel^2 \leq (\mathcal{M}-1)\vartheta c_1$, and

$$\sum_{i=1}^{\mathcal{M}} \parallel y_{II}^i(t) \parallel^2 \leq a + \beta_2(\mathcal{M}-1)\vartheta c_1 := a + b.$$

b) Case $t \in [\delta^1, 2\delta^1]$, then $t - \delta_{ij}$ belongs to either $[-\delta, 0]$ or $[0, \delta^1]$, we get

$$\sum_{i=1}^{\mathcal{M}} \parallel y_{II}^i(t) \parallel^2 \leq a + \beta_2\Big[(\mathcal{M}-1)\vartheta c_1 + (\mathcal{M}-1)(a+b)\Big] := \Big[1 + \beta_2(\mathcal{M}-1)\Big](a+b).$$

c) Case $t \in [0; (k+1)\delta^1] \cap [0, L]$; $k\delta^1 \leq L, k = 0, 1, ...,$ we get

$$\sum_{i=1}^{\mathcal{M}} \parallel y_{II}^i(t) \parallel^2 \leq \sum_{l=0}^{\mathcal{M}} \Big[\beta_2(\mathcal{M}-1)\Big]^l (a+b).$$

Thus, for $t \in [0, L]$, we have

$$\sum_{i=1}^{\mathcal{M}} \parallel y_{II}^i(t) \parallel^2 \leq \sum_{l=0}^{[\frac{L}{\delta^1}]} \Big[\beta_2(\mathcal{M}-1)\Big]^l (a+b) := \vartheta_5(a+b). \tag{18}$$

Thus, we have

$$y(t)^\top R y(t) \leq \vartheta_2[\sum_{i=1}^{\mathcal{M}} \parallel y_I^i(t) \parallel^2 + \sum_{i=1}^{\mathcal{M}} \parallel y_{II}^i(t) \parallel^2] \leq \vartheta_2[\vartheta_4 e^{\beta L} + \vartheta_5(a+b)] < c_2,$$

which completes the proof. ■

Remark 1 In theorem 1, we used singular value theory to analyze the singular system to slow and fast subsystems and constructed appropriated Lyapunov-Krasovskii functionals to get an estimate of the slow subsystem and perturbation approach to investigate the boundedness of solutions of the fast subsystem. Although the parameter $\beta > 0$ is not a linear variable in conditions [26], however, the parameter is not involved in LMI condition (8), then we first find the solutions of LMI (8) satisfying (7), (9) and then check the condition (10). To solve the LMI (8), we can utilize Matlab LMI Control Toolbox.

Remark 2 This paper considers linear singular LSNNs that contain interacted delay terms among all subsystems. If the considered system becomes a regular system ($E = I$), the stability conditions obtained in this paper can be reduced to the stability conditions for normal large-scale neural networks [7, 10]. For the descriptor large-scale with delays, the stability conditions of Theorem 1 can be considered as an extension of the results of [22-24], where the neural structure is not considered.

Remark 3 The robust finite-time stability conditions for singular LSNNs can be performed by the following the procedure.

1. Give some fixed parameters c_1, L, h and $R_i, i = 1, 2, ..., \mathcal{M}$.

2. Provide an initial scalar c_2.

3. Initiating from stable scalar $\beta > 0$, we kept raising $c_2 > 0$ until we find a solution.

4. If the issue is infeasible, then the initial value c_2 must be raised. Otherwise, c_2 can be can be reduced till it reaches its minimum.

Example 1 Consider system (1), where $\mathcal{M} = 3$ and

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2.8 \end{bmatrix}; A_2 = \begin{bmatrix} 3.8 & 0 \\ 0 & 3 \end{bmatrix}; A_3 = \begin{bmatrix} 3 & 0 \\ 0 & 3.7 \end{bmatrix}; B_1 = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.9 \end{bmatrix}; B_2 = \begin{bmatrix} 1 & 0.1 \\ 1 & 0.8 \end{bmatrix};$$

$$E_1 = E_2 = E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; B_3 = \begin{bmatrix} 1.9 & 1 \\ 0 & 0.5 \end{bmatrix}; C_{12} = \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.7 \end{bmatrix};$$

$$C_{13} = \begin{bmatrix} 1.2 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}; C_{21} = \begin{bmatrix} 1.5 & 0 \\ 0.5 & 0.1 \end{bmatrix}; C_{23} = \begin{bmatrix} 0.5 & 0.6 \\ 0.9 & 0.3 \end{bmatrix};$$

$$C_{31} = \begin{bmatrix} 0 & 0.1 \\ 0.8 & 0.6 \end{bmatrix}; C_{32} = \begin{bmatrix} 1 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}; D_1 = \begin{bmatrix} 0.11; 0.6 \end{bmatrix}; D_2 = \begin{bmatrix} -0.5; 0.5 \end{bmatrix}; D_3 = \begin{bmatrix} 0.2; 0.6 \end{bmatrix};$$

$$U_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}; U_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}; U_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}; V_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix};$$

$$V_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}; V_3 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}; R_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}; R_2 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix};$$

$$R_3 = \begin{bmatrix} 0.29 & 0 \\ 0 & 0.28 \end{bmatrix}; \delta = \max\{\delta_{ij}\} = 0.5, \delta^1 = \min\{\delta_{ij}\} = 0.1, h = 1.$$

Taking $\beta = 0.01, c_1 = 0.1, c_2 = 11.1, L = 10$, and with the help of Matlab LMI Control Toolbox, the LMI (8) is feasible with solutions:

$$P_1 = \begin{bmatrix} 5.1738 & -1.3180 \\ 0 & 3.4260 \end{bmatrix}; P_2 = \begin{bmatrix} 5.1206 & -0.0607 \\ 0 & 5.2557 \end{bmatrix}; P_3 = \begin{bmatrix} 4.5285 & -0.6665 \\ 0 & 4.7688 \end{bmatrix};$$

$$X_1 = \begin{bmatrix} 0.4784 & -0.0375 \\ -0.0715 & 0.2944 \end{bmatrix}; X_2 = \begin{bmatrix} 0.4629 & -0.0842 \\ -0.1451 & 0.5900 \end{bmatrix}; X_3 = \begin{bmatrix} 0.3452 & 0.1748 \\ 0.0554 & 0.2701 \end{bmatrix};$$

$$Y_1 = \begin{bmatrix} 1.4426 & -0.3841 \\ -0.3841 & 1.4530 \end{bmatrix}; Y_2 = \begin{bmatrix} 3.3653 & 0 & -0.0037 \\ -0.0037 & 3.0247 & \end{bmatrix}; Y_3 = \begin{bmatrix} 1.5029 & -0.6926 \\ -0.6926 & 2.5908 \end{bmatrix};$$

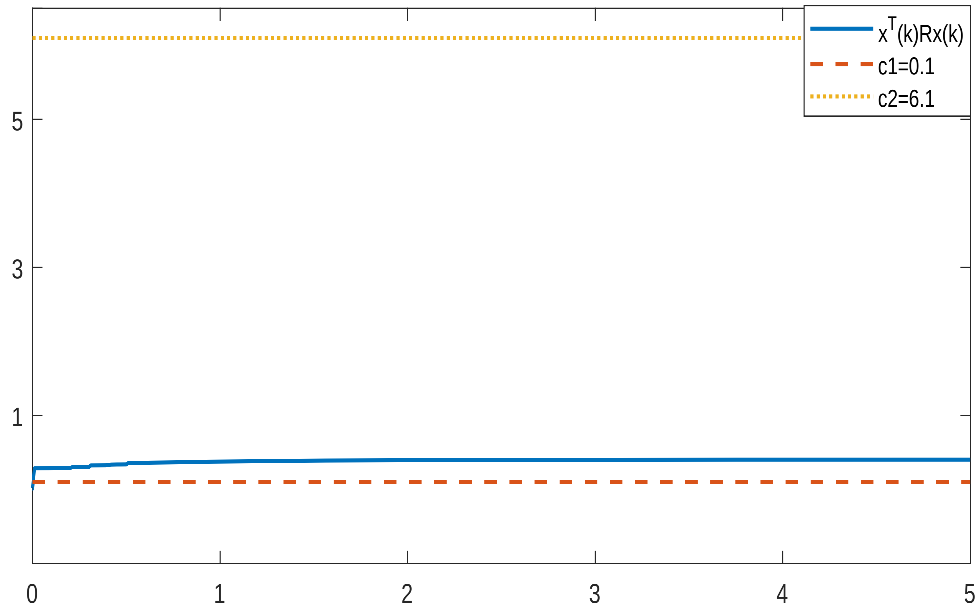

Fig. 1 The time history of $x^\top(t)Rx(t)$ for the system

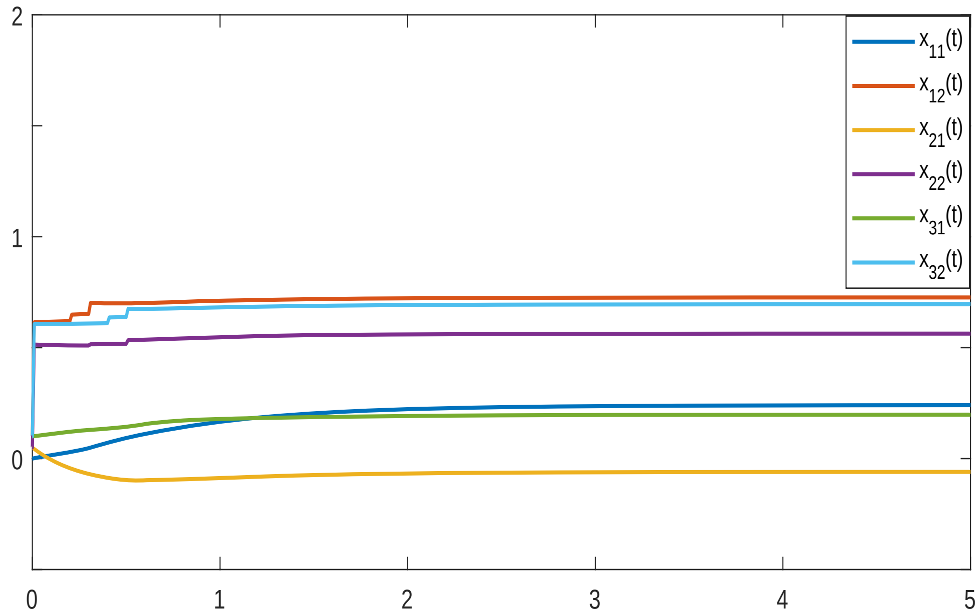

Fig. 2 State responses of the subsystems

$$Q_1 = \begin{bmatrix} 4.2403 & -1.2844 \\ -1.2844 & 3.3784 \end{bmatrix}; \; Q_2 = \begin{bmatrix} 7.7610 & -0.0086 \\ -0.0086 & 6.1456 \end{bmatrix}; \; Q_3 = \begin{bmatrix} 5.9546 & -1.2119 \\ -1.2119 & 5.9138 \end{bmatrix};$$

$$Z_1 = \begin{bmatrix} 9.6444 & 0 \\ 0 & 6.1795 \end{bmatrix}; \; Z_2 = \begin{bmatrix} 11.7862 & 0 \\ 0 & 9.6367 \end{bmatrix}; \; Z_3 = \begin{bmatrix} 10.3435 & 0 \\ 0 & 11.0326 \end{bmatrix};$$

$$S_1 = \begin{bmatrix} 0.9192 & -0.4151 \\ -0.4151 & 0.4621 \end{bmatrix}; \; S_2 = \begin{bmatrix} 0.3718 & 0.0135 \\ 0.0135 & 0.4370 \end{bmatrix}; \; S_3 = \begin{bmatrix} 0.3276 & -0.2416 \\ -0.2416 & 0.4874 \end{bmatrix};$$

Figure 1 and Figure 2 describe the time history of $x^\top Rx(t)$ and the solution response of the system with the initial functions $\phi(t) = [\phi_1(k),\ \phi_2(k),\ \phi_3(k)]$, $\phi_1(k) = [0.1\sin(t)e^t, 0.1]$, $\phi_2(k) = [0.05e^t, 0.05e^t]$, $\phi_3(k) = [0.1e^t, 0.1]$, respectively.

4 Conclusions

The robust FTS for singular LSNNs with interconnected delays has been investigated in this paper. Based on the singular value theory and Lyapunov function method combined with LMI technique, new delay-dependent sufficient conditions for the robust FTS have been established via solving tractable LMIs. The effectiveness and validity of the obtained results are illustrated y a numerical example. The suggested technique can be extended to the situation of singular LSNNs, where the delays of the system are time-varying.

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