Is Mathematics unreasonable effective? Why?
Mathematical versus physical reality
Are black holes real?

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INTRODUCTION

Philosophical questions:

1. Is Mathematics unreasonable effective? How and Why?
2. Is mathematics a Science? Invention or discovery?

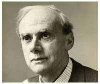
Will probe this question from the perspective of General Relativity

- The mathematical framework of GR.
- Black holes as mathematical objects.
- On the reality of black holes. A mathematical perspective.
- Stability of slowly rotating black holes.

E. WIGNER

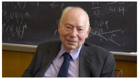
"Unreasonable Effectiveness of Mathematics in the Natural Sciences" (1960).

- "Science of skillful operations with concepts and rules invented just for this purpose".
- The purpose is for mathematicians to "demonstrate [their] ingenuity and sense of formal beauty.
- Mathematical concepts "are defined with a view of permitting ingenious logical operations which appeal to our aesthetic sense" ... "chosen for their amenability to clever manipulations and to striking, brilliant arguments".



C. Dirac 1939

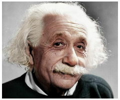
" ... the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen".



S. Weinberg 1996

“It is positively spooky how the physicist finds the mathematician has been there before him or her. Physicists generally find the ability of mathematicians to anticipate the mathematics needed in the theories of physics quite uncanny.

It is as if Neil Armstrong in 1969 when he first set foot on the surface of the moon had found in the lunar dust the footsteps of Jules Verne”.

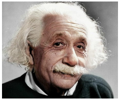


1934 MATHEMATICS/EXPERIENCE

- All knowledge about reality begins with experience and terminates in it.
- Experience can guide us in our choice of serviceable mathematical concepts; it cannot be the source from which they are derived.

THE TRULY CREATIVE PRINCIPLE RESIDES IN MATH.

- Pure mathematical construction enables us to discover the concepts and the laws connecting them which give us the key to the understanding of the phenomena of Nature.
- In a certain sense, therefore, I hold it to be true that pure thought is competent to comprehend the real, as the ancients dreamed.



MATHEMATICAL FRAMEWORK OF PHYSICS

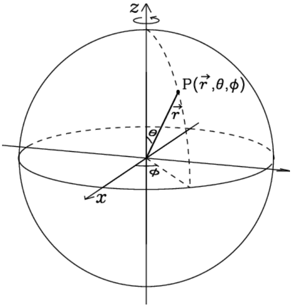
- The physical world is represented as a four-dimensional continuum. If in this I adopt a Riemannian metric, and look for the simplest laws which such a metric can satisfy, I arrive at the relativistic gravitation theory of empty space.
- If I adopt in this space a vectorfield, or in other words, the anti-symmetrical tensor-field derived from it, and if I look for the simplest laws which such a field can satisfy, I arrive at the Maxwell equations for free space.

WHAT DID EINSTEIN MEAN?

The physical world is represented as a four ...

- Euclidean space. $\mathbb{R}^3$
- Surfaces. $f(x, y, z) = 0,\ df \neq 0$
- Parametrizations.
- Change of coordinates. Rigid motions in $\mathbb{R}^3$.
- Manifold
- Metric
- Tensors
- Derivatives of tensors -Levi Civitta connection
- Curvature
- Ricci curvature
- Einstein Field Equations
- Differential Equations.

Parametrizations

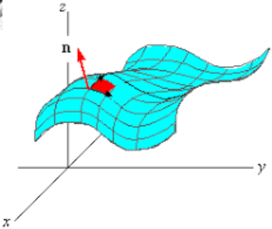


Sphere $\mathbb{S}^2$. $x^2 + y^2 + z^2 = 1.$

Parametrizations:

$$x = u, \quad y = v, \quad z = \pm\sqrt{1 - u^2 - v^2}$$

$$x = \cos\theta, \quad y = \sin\theta\cos\varphi, \quad z = \sin\theta\sin\varphi$$



$$\vec{X}(u,v) = \left(X^1(u,v), X^2(u,v), X^3(u,v)\right).$$

$$\vec{X}_u(u,v), \quad \vec{X}_v(u,v).$$

Normal. $\vec{N}(u,v) = \pm\frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}.$

Curvature. $|K(p)| = \frac{|\vec{N}_u \times \vec{N}_v|}{|\vec{X}_u \times \vec{X}_v|}.$

Metric $g = Edu^2 + 2Fdudv + Gdv^2.$

$$E = \vec{X}_u \cdot \vec{X}_u, \quad F = \vec{X}_u \cdot \vec{X}_v, \quad G = \vec{X}_v \cdot \vec{X}_v$$

T. Egregium. K depends only on the intrinsic geometry of S.

RIEMANNIAN GEOMETRY

- Manifold M^n. Coordinate charts $(x^1, \dots x^n)$.
- Coordinate transformations $(x^1, \dots x^n) \longrightarrow (x'^1, \dots x'^n)$.
- Vectors. $X^1\partial_1 + X^2\partial_2 + \dots X^n\partial_n$, $(X^1, \dots, X^n)$.
- 1-forms. $w_1 dx^1 + w_2 dx^2 + \dots w_n dx^n$, $(w_1, \dots, w_n)$.
- Tensors. T_{ab}, T_a^b, T^{ab}
- Metric. $g = g_{ab} dx^a dx^b$.

  Einstein convention. $T^{ab} = g^{as} g^{bt} T_{st}$.

RIEMANNIAN GEOMETRY

- Covariant derivatives. $X_a \to D_b X^a = \partial_b X^a - \Gamma^a_{bc} X^c$
- Covariant derivatives do not commute

$$D_a D_b X^c - D_b D_a X^c = R_d{}^c{}_{ab} X^c$$

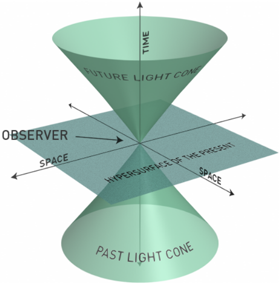
- Riemann curvature tensor $R_d{}^c{}_{ab}$.
- Symmetries. $R_{abcd} = R_{cdab} = \mathbf{R}_{bacd}$
- Bianchi Identities

$$\begin{aligned} R_{abcd} + R_{acdb} + R_{adbc} &= 0, \\ D_s R_{abcd} + D_a R_{bscd} + D_b R_{sacd} &= 0. \end{aligned}$$

- Ricci curvature tensor R_{ab}.

SPECIAL RELATIVITY

MINKOWSKI SPACE $\mathbb{R}^{1+3}$,



$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$

Lorentz transformation

Electromagnetic field. $F = dA, \quad A = A_\alpha dx^\alpha.$

Maxwell equations. $dF = 0, \quad \delta F = 0.$

Wave equation $\Box\phi = 0.$

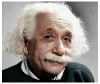
WAVE EQUATION IN $\mathbb{R}^{1+n}$ $\quad \Box\psi = 0$

- **Energy conservation.** $\mathcal{E}_s(t) = \mathcal{E}_s(0)$.
- **Pointwise Bound.** $\|\partial\psi(t)\|_\infty \lesssim \mathcal{E}_s(0), \quad s > \frac{n}{2}$,
- **Pointwise Decay.** $\|\partial\psi(t)\|_\infty \lesssim t^{-\frac{n-1}{2}}$.
- **V-field method.** $[\Box, \Gamma] \sim \Box, \quad e_\pm = \partial_t \pm \partial_r,\ \underline{u}, u = t \pm r$.

$$|\psi(t,x)| \lesssim (1+|\underline{u}|)^{-\frac{n-1}{2}}(1+|u|)^{-1/2}\|\Gamma^{\frac{n}{2}+1}\psi(0)\|_{L^2}$$

$$\begin{aligned} |(\nabla, e_+)\psi(t,x)| &\lesssim (1+|\underline{u}|)^{-\frac{n-1}{2}-1}(1+|u|)^{-\frac{1}{2}}\|\Gamma^{\frac{n}{2}+1}\psi(0)\|_{L^2} \\ |e_-\ \psi(t,x)| &\lesssim (1+|\underline{u}|)^{-\frac{n-1}{2}}(1+|u|)^{-\frac{1}{2}-1}\|\Gamma^{\frac{n}{2}+1}\psi(0)\|_{L^2} \end{aligned}$$

- **Null condition.** Algebraic structure of nonlinear terms.



GENERAL RELATIVITY

LORENTZIAN METRICS. $g = g_{\alpha\beta} dx^\alpha dx^\beta, \quad -+++.$

EQUIVALENCE PRINCIPLE $\mathbf{g} \equiv \phi^* \mathbf{g}, \quad \phi : \mathbf{M} \longrightarrow \mathbf{M}.$

FIELD EQUATIONS. $\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta} = \mathbf{T}_{\alpha\beta}$

BIANCHI. $\mathbf{D}^\beta \mathbf{R}_{\alpha\beta} = 0 \Rightarrow \mathbf{D}^\beta \mathbf{T}_{\alpha\beta} = 0$

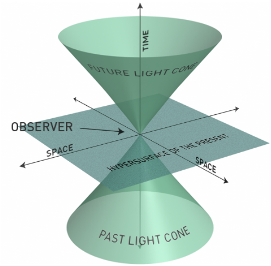
Spacetime tells matter how to move; matter tells spacetime how to curve.

EINSTEIN-MAXWELL. $\mathbf{T}_{\alpha\beta} = \mathbf{F}_{\alpha\lambda}\mathbf{F}_{\beta}{}^{\lambda} - \frac{1}{4}\mathbf{g}_{\alpha\beta}\left(F_{\lambda\mu}\mathbf{F}^{\lambda\mu}\right)$

VACUUM EQTS. $\mathbf{R}_{\alpha\beta} = 0.$

GEOMETRIC FRAMEWORK OF GR

CAUSALITY LORENTZIAN MANIFOLDS (M, g)



- Inertia–$T_p(M)$ = Minkowski
- Events≡ points in M
- Observers≡ timelike curves
- Geodesics≡ free moving
- Light rays≡ null geodesics
- EP ≡ Gen. covariance
- Tidal forces ≡ curvature
- Isolated system≡ A. flat

GEODESICS $\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0.$

HYPERSURFACES $f = const, \quad \mathbf{D}f \neq 0.$

- spacelike
- null $\mathbf{g}^{\alpha\beta}\partial_\alpha f \partial_\beta f = 0,$ characteristics
- timelike

GEOMETRIC FRAMEWORK

1. Null Pair (e_3, e_4), $\mathbf{g}(e_3, e_4) = -2$.

2. Horizontal structure $\mathcal{H} := \{e_3, e_4\}^\perp$.

3. Connection coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \omega, \underline{\omega}$.

$$
\begin{aligned}
\chi_{ab} &= \mathbf{g}(\nabla_a e_4, e_a), \qquad \underline{\chi}_{ab} = \mathbf{g}(\nabla_a e_3, e_b) \\
\chi_{ab} &= \hat{\chi}_{ab} + \frac{1}{2}\mathrm{tr}\,\chi\delta_{ab} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\chi \in_{ab} \\
\underline{\chi}_{ab} &= \underline{\hat{\chi}}_{ab} + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\delta_{ab} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\underline{\chi} \in_{ab}
\end{aligned}
$$

$${}^{(a)}\mathrm{tr}\,\chi = {}^{(a)}\mathrm{tr}\,\underline{\chi} = 0 \Rightarrow \quad \text{Integrability.}$$

4. Curvature coefficients $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, {}^*\rho$

$$\alpha_{ab} = \mathbf{R}(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} = \mathbf{R}(e_a, e_3, e_b, e_3)$$

GEOMETRIC FRAMEWORK

5. Complete set of null decompositions.
   - Connection $\Gamma = \left\{\chi,\ \underline{\chi},\ \xi,\ \underline{\xi},\ \eta,\ \underline{\eta},\ \zeta, \omega,\ \underline{\omega},\ \right\}$
   - Curvature $R = \{a,\ \rho,\ {}^*\rho, \underline{\beta}\,, \underline{\alpha}\}$

6. Cartan-Bianchi. Tensorial character!

$$d\Gamma + [\Gamma, \Gamma] \quad = \quad R, \qquad\quad dR + [R, \Gamma] = 0.$$

7. Null frame transformations

$$(e_3, e_4, \mathcal{H}) \to (e_3', e_4', \mathcal{H}'), \qquad\quad (\Gamma, R) \to (\Gamma', R')$$

MATHEMATICS OF GR

- Topological arguments. Causality
- ODEs.
- First order scalar eqts. $\mathbf{g}^{\alpha\beta}\partial_\alpha f \partial_\beta f = 0.$
- Transport eqts.
- Elliptic eqts. Induced on spacelike hypersurfaces.
- Hyperbolic eqts. Bianchi Identities

HYPERBOLIC EQTS. Wave coordinates x^α, $\square_g x^\alpha = 0.$

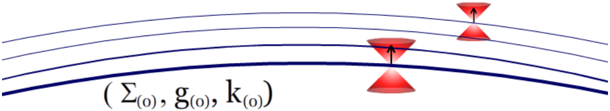
$$\mathbf{R}_{\alpha\beta} = 0 \Rightarrow \boxed{g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = N_{\alpha\beta}(g, \partial g)}$$

INITIAL VALUE FORMULATION

HYPERBOLICITY. Wave coordinates x^α, $\Box_{\mathbf{g}} x^\alpha = 0.$

$$\mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu \mathbf{g}_{\alpha\beta} = N_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g})$$

INITIAL DATA SETS. $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ $+$ Constraints



THEOREM(Bruhat-Geroch) *Smooth IDS admit unique, smooth, maximal future globally hyperbolic developments (MFGHD).*

BLACK HOLES

CHANDRASEKHAR: “macroscopic objects with masses varying from a few solar masses to millions of solar masses. ...they are all, every single one of them, described exactly by the Kerr solution.

This is the only instance we have of an exact description of a macroscopic object.

Macroscopic objects, as we see them around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time.

They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single two parameter family of solutions for their description, they are the **simplest** as well.”

KERR FAMILY $\mathcal{K}(a, m)$

$$-\frac{|q|^2\Delta}{\Sigma^2}(dt)^2+\frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\varphi-\frac{2amr}{\Sigma^2}dt\right)^2+\frac{|q|^2}{\Delta}(dr)^2+|q|^2(d\theta)^2$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ |q|^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2+a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

STATIONARY, AXISYMMETRIC. $\partial_t, \ \partial_\varphi$ KILLING

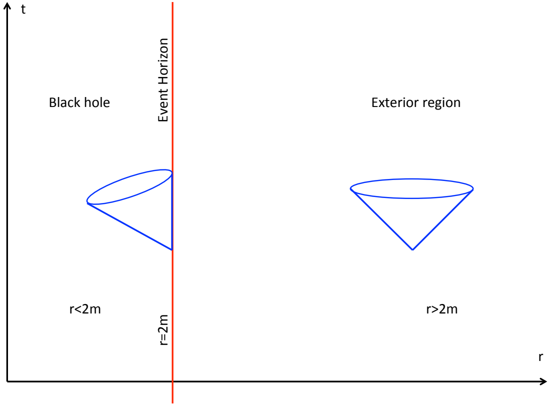
ASYMPTOTICALLY FLAT. Approaches Minkowski as $r \to \infty$.

SCHWARZSCHILD. $a = 0, m > 0,$ sph. symmetric.

$$-\frac{\Delta}{r^2}(dt)^2+\frac{r^2}{\Delta}(dr)^2+r^2d\sigma_{\mathbb{S}^2}, \qquad \frac{\Delta}{r^2}=1-\frac{2m}{r}$$

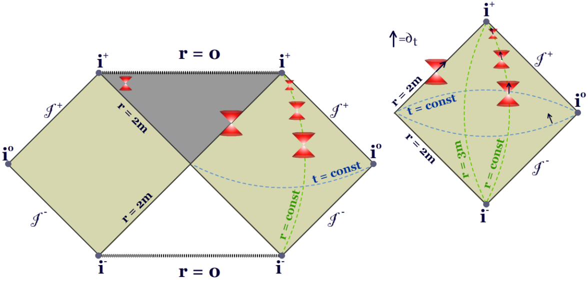
SCHWARSZCHILD (1916) BLACK HOLE

$$-\frac{\Delta}{r^2}(dt)^2 + \frac{r^2}{\Delta}(dr)^2 + r^2 d\sigma_{\mathbb{S}^2}, \qquad \frac{\Delta}{r^2} = 1 - \frac{2m}{r}$$



SCHWARSZCHILD. MAXIMALLY EXTENDED

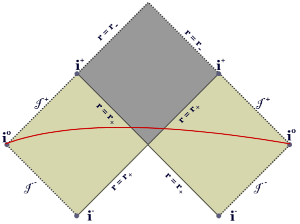
$$-\frac{\Delta}{r^2}(dt)^2 + \frac{r^2}{\Delta}(dr)^2 + r^2 d\sigma_{\mathbb{S}^2}, \qquad \frac{\Delta}{r^2} = 1 - \frac{2m}{r}$$



- **EXTERNAL REGION** $r > 2m$
- **EVENT HORIZON** $r = 2m$
- **BLACK HOLE** $r < 2m$
- **SINGULARITY** $r = 0$
- **NULL INFINITY** $r = \infty$.

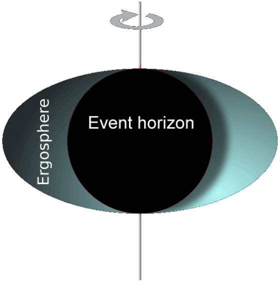
KERR SPACETIME $\mathcal{K}(a, m)$, $|a| \leq m$

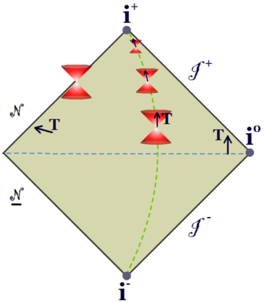
MAXI. EXTENSION $\Delta(r_-) = \Delta(r_+) = 0$, $\Delta = r^2 + a^2 - 2mr$



- **EXTERNAL REGION** $r > r_+$
- **EVENT HORIZON** $r = r_+$
- **BLACK HOLE** $r < r_+$
- **CAUCHY HORIZON** $r < r_+$
- **NULL INFINITY** $r = \infty$

EXTERNAL KERR



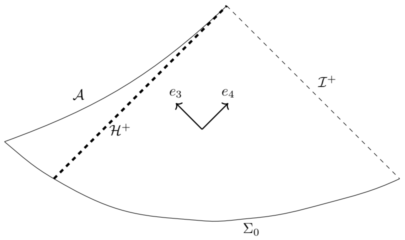


- **ERGOREGION** $\quad \mathbf{g}(T, T) < 0$
- **TRAPPED NULL GEODESICS**

PRINCIPAL NULL PAIR. $\{e_3, e_4\}$

- Diagonalizes the curvature tensor.
- Horizontal structure $\mathcal{H} = \{e_3, e_4\}^\perp$ is non-integrable if $a \neq 0$.

$$\underline{\chi}(e_a, e_b) = \mathbf{g}(\mathbf{D}_{e_a} e_3, e_b), \qquad \chi(e_a, e_b) = \mathbf{g}(\mathbf{D}_{e_a} e_4, e_b)$$



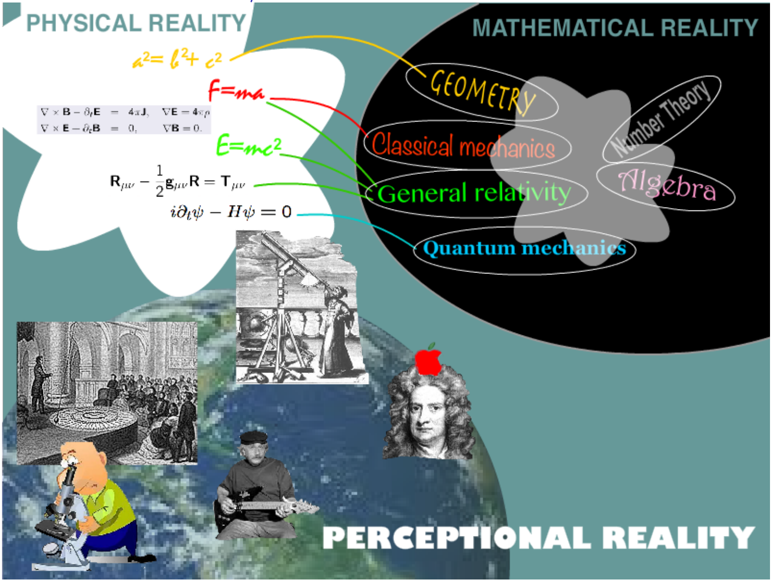
ARE BLACK HOLES REAL ?

Is an object **physically** real even when we don't have **in principle** direct access to it ?

MATHEMATICAL REALITY. Object is **real** if mathematical **self-consistent**.

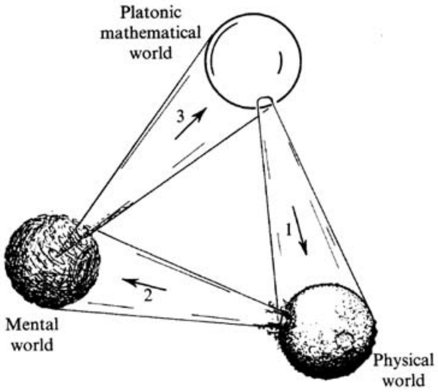
PHYSICAL REALITY. Object is **real** if it leads to **by inference**, in the framework of an **acceptable mathematical theory**, to observable, measurable, effects.

MATHEMATICAL/ PHYSICAL REALITY



CAN WE TEST PHYSICAL REALITY WITHIN MATHEMATICS?

PENROSE: THREE WORLDS PICURE OF REALITY

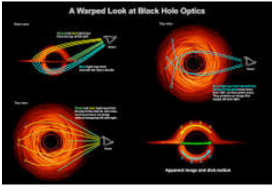


ARE BLACK HOLES REAL ? 3 NOBEL PRIZES

Is an object **physically** real even if it is **undetectable** ?

- Astrophysical observations. R. Gentzel, A. Ghez
- Gravitational wave detectors. B. Barish, K. Thorne, R. Weiss
- Mathematical theorems R. Penrose
- Numerical simulations F. Pretorius

Accretion disk



PRETORIUS

- Black Hole(BH)-one of the most fascinating and enigmatic predictions of Einstein's theory of general relativity.
- Interior - rich structure and is intrinsically dynamical, where space and time itself are inexorably led to a singular state.
- Exterior- remarkably simple, described uniquely by the stationary Kerr solution.
- BHs- Though "discovered" purely through thought and the mathematical exploration of a theory far removed from every day experience, they appear to be ubiquitous in our universe.

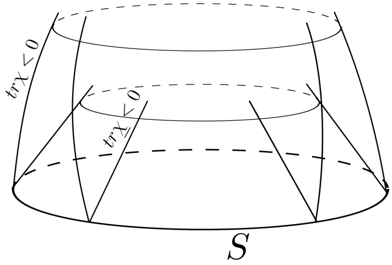
ASTROPHYSICAL EVIDENCE

- Consistent with the high luminosity of quasars.
- Consistent with the effects of several dozen X-ray binary systems, too massive to be neutron stars.
- Consistent with the dynamical motion of stars and gas about the centers of nearby galaxies and our Milky Way Galaxy - supermassive black holes.

PENROSE SINGULARITY THEOREM

THEOREM. Space-time (M, g) cannot be future null geodesicaly complete, if

- $\mathrm{Ric}(g)(L, L) \geq 0, \qquad \forall L$ *null*
- M contains a non-compact Cauchy hypersurface
- M contains a closed **trapped** surface S



Null expansions $\quad \mathrm{tr}\,\chi, \qquad \mathrm{tr}\,\underline{\chi}$

WHAT IS MATHEMATICAL GR ?

1. ELUCIDATE THE MATHEMATICAL STRUCTURE OF CLASSICAL GR

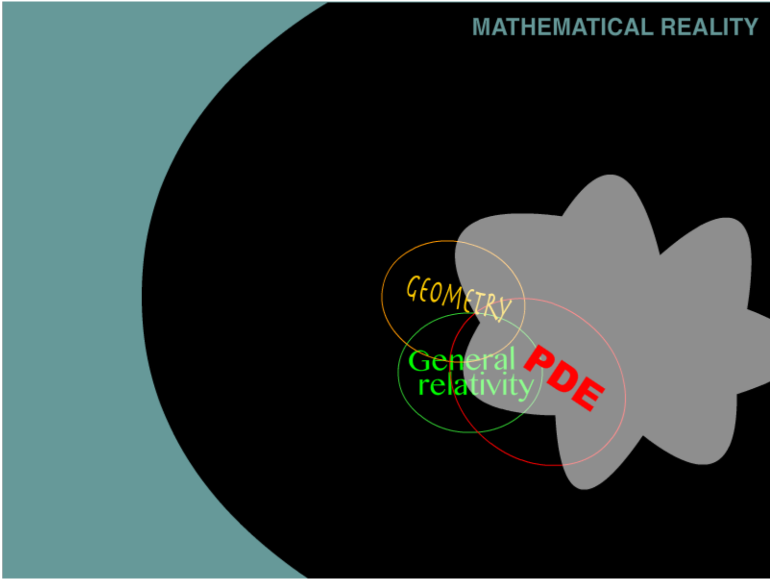
2. FORMULATE AND ADDRESS ITS CENTRAL PROBLEMS.
   - PHYSICALLY RELEVANT
   - SATISFIES OUR MATHEMATICAL SENSIBILITIES: Rigor, Mathematical Challenges, Formal Beauty.

3. ESTABLISH CONECTIONS TO OTHER PROBLEMS PDE, GEOMETRY, MATH. PHYSICS ...

MATHEMATICAL ENTANGLEMENT

Math concepts introduced for solving specific problems have unexpected, mysterious, consequences in seemingly unrelated areas

MATHEMATICAL RELATIVITY



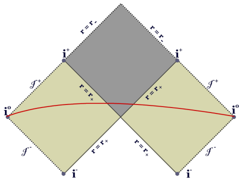
MATHEMATICAL TESTS OF REALITY

1. COLLAPSE. Can black holes (trapped surfaces) form starting with reasonable initial data configurations?

2. RIGIDITY. Does the Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, exhaust all possible stationary solutions?

3. STABILITY. Is the Kerr family stable under, general, arbitrary small perturbations?

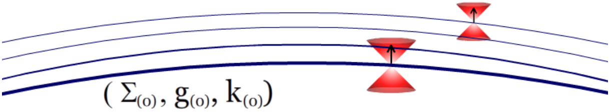
4. COSMIC CENSORSHIP



FINAL STATE CONJECTURE

The evolution of generic initial data sets behave, for large time, like a finite number of Kerr black holes, moving away from each other, plus a radiative decaying term.

1. Small data don't concentrate, i.e. lead to pure, *slowly decaying*, gravitational waves. Stability of Minkowski space.
2. Large data may concentrate to produce stationary states, i.e. BHs. Collapse.
3. All stationary states are Kerr. Rigidity.
4. Kerr solutions are stable. Stability.



FINAL STATE CONJECTURE

5. There can be no singularities outside BHs. Cosmic Censorship Conjecture.

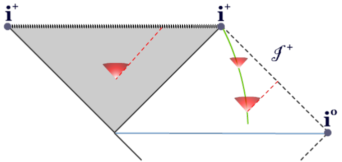


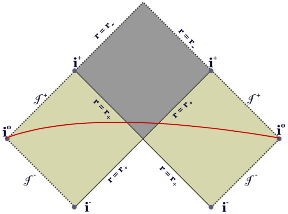
Figure: Null geodesics outside and inside the black hole

6. Two (and more) body problem.

STABILITY OF KERR

CONJECTURE[Stability of (external) Kerr].

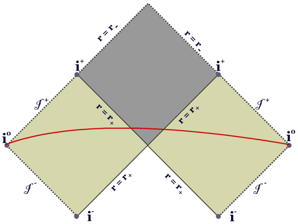
Small perturbations of a given exterior Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.



THEOREM “True” if $|a|/m \ll 1$.

- MAIN[K-Szeftel(2021)]
- GCM PAPERS[K-Szeftel(2019), Shen(2022)]
- WAVE PAPER[Giorgi-K-Szeftel(2022)]

MAIN DIFFICULTIES I



- Strongly coupled, tensorial, nonlinear character of EVE
- Gauge group= All diffeomorphisms $\quad \mathbf{g} \equiv \Phi^* \mathbf{g}$.
- Slow convergence. Decay.
- Nontrivial nature of Kerr. Ergoregion, Trapping, Null ∞ Non-integrability.

MAIN DIFFICULTIES II

- What are the final parameters?
- Convergence to final state is gauge dependent!
- Final parameters and correct gauge emerge in the limit!
- Low rates of time decay to the final state. Integrable quantitative decay. Dispersion.

SUMARY

According to common opinion:

- Mathematical concepts are creations of great mathematicians and originate in aesthetic preferences.
- Mathematics does not deal with objective reality and therefore is not a science!

In fact

- Mathematics deals with its own form of reality. It proceeds not by invention but advances by a natural process of probing and discovering new territories.
- Though aesthetic preferences play an important role, they are not the determining factors.

CONCLUSIONS

Moreover

- In probing the physical world the truly creative principle resides in mathematics.
- Concepts introduced for solving specific problems turn out to have unexpected and mysterious consequences in seemingly unrelated areas. **Mathematical Entanglement.**
- Mathematical GR provides a great example of both of the above